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Resurgent Deformations for an Ordinary Differential Equation of Order 2.

Eric Delabaere[*], *Jean-Marc Rasoamanana* [†]

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Abstract

We consider in the complex field the differential equation $\frac{d^2}{dx^2}\Phi(x) = \frac{P_m(x,\underline{a})}{x^2}\Phi(x)$, where P_m is a monic polynomial function of order m with coefficients $\underline{a} = (a_1, \cdots, a_m)$. We investigate the asymptotic, resurgent, properties of the solutions at infinity, focusing in particular on the analytic dependence on $\underline{a}$ of the Stokes-Sibuya multipliers. Taking into account the non trivial monodromy at the origin, we derive a set of functional equations for the Stokes-Sibuya multipliers. We show how these functional relations can be used to compute the Stokes multipliers for a class of polynomials P_m. In particular, we obtain conditions for isomonodromic deformations when $m = 3$.

1 Introduction

This article is the first of a series of three papers to come. The motivation stems from the well-known theory of Sibuya [35] and its Gevrey-resurgent extensions, and their applications in spectral analysis.

In [35], Sibuya gives an exhaustive description of the asymptotic properties when $|x| \to \infty$ of the solutions of the ordinary differential equation $-\frac{d^2\Phi}{dx^2} + P(x)\Phi = 0$, where $P(x) = x^m + a_1x^{m-1} + \cdots + a_m$ is a complex polynomial function of order m. Among various results, he shows the existence of a set of fundamental functional relations between the Stokes connection matrices, when viewed as functions of the coefficients of P. The asymptotic behavior of the Stokes-Sibuya coefficients when the constant term a_m of P tends to infinity is also provided. Later, these results have been clarified and extended in the framework of the Gevrey and resurgence theories [31, 25, 20, 5, 40, 9, 12, 7, 38].

One of the main applications of the above results is both the qualitative and quantitative description of the spectral set of the Schrödinger operator $-\frac{d^2}{dx^2} + P(x)$, e.g., [10, 11, 24, 41]. Recently, the exact asymptotic analysis has been applied with success to describe the spectral

[*]Département de Mathématiques, UMR CNRS 6093, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers Cedex 01, France / tel. +33 (0)2.41.73.54.94 / fax. +33 (0)2.41.73.54.54 / e-mail. eric.delabaere@univ-angers.fr

[†]Département de Mathématiques, UMR CNRS 6093, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers Cedex 01, France / tel. +33 (0)2.41.73.54.55 / fax. +33 (0)2.41.73.54.54 / e-mail. jean-marc.rasoamanana@univ-angers.fr

properties of a class of PT-symmetric oscillators, i.e., when the potential function P satisfies $P(x) = \overline{P(-\overline{x})}$. As a rule, such PT-symmetric operators are not Hermitian, therefore the existence of a real spectrum is a non obvious but interesting question (see, e.g., [2, 3] for the motivations and applications in physics). In [14, 15], the authors consider the PT-symmetric complex cubic oscillator, and prove the reality of the spectrum and the appearance of spontaneous symmetry breaking, this depending on the values of the coupling constant. Meanwhile, the reality of the spectrum of the PT-symmetric Schrödinger operator $-\dfrac{d^2}{dx^2} + P(x)$ with P polynomial was proved by Shin [34] under convenient hypotheses. His work relies on a clever examination of the results of Sibuya, and on ideas and tools usually used in the context of integrable models in quantum field theory.

As a matter of fact, apart from the key-results of Sibuya, the strategy followed by Shin was previously developed by Dorey et al [17] to prove the reality of the spectrum of the PT-symmetric operator $-\dfrac{d^2}{dx^2} - (ix)^{2m} - \alpha(ix)^{m-1} + \dfrac{l(l+1)}{x^2}$.

Our programme is to generalize all the above results to the one dimensional Schrödinger operator $H = -\dfrac{d^2}{dx^2} + \dfrac{P(x)}{x^2}$ with $P(x)$ a complex polynomial function of order $m \in \mathbb{N}^\star$.

In the present article, we consider the ordinary differential equation

$$(\mathfrak{E}_m) \qquad \frac{d^2}{dx^2}\Phi(x) = \frac{P_m(x,\underline{a})}{x^2}\Phi(x)$$

in the complex x plane, where $\underline{a} := (a_1, \cdots, a_m) \in \mathbb{C}^m$, $m \in \mathbb{N}^\star$, and :

$$P_m(x,\underline{a}) = x^m + a_1 x^{m-1} + \ldots + a_m \in \mathbb{C}[x].$$

This equation admits a unique irregular singular point located at infinity, and our aim is to concentrate on the asymptotic behaviors of the solutions of $(\mathfrak{E}_m)$ at this point, and their deformations in the parameter $\underline{a} = (a_1, \ldots, a_m)$. Compared with Sibuya's work [35], the main novelty comes from the existence (as a rule) of another singular point, a regular singular one at the origin. The Stokes-Sibuya coefficients, when considered as functions of the coefficients $\underline{a}$ of the polynomial P_m, are still governed by a set of independent functional relations, but the non trivial monodromy at the origin has now to be taken into account. As we shall see, this translates in term of an interesting $\underline{a}$-dependent equational resurgence structure.

The paper is organized as follows. In section 2 we study the solutions near infinity, introducing a well-behaved family of systems of fundamental solutions. The associated Stokes-Sibuya coefficients are defined, and their analytic dependence on $\underline{a}$ are analyzed. The main existence theorem given in 2 is proved in section 3 by resurgent methods, and we compare the Stokes-Sibuya coefficients with the Stokes multipliers given by the resurgence viewpoint.

In section 4, we introduce and study a convenient system of fundamental solutions near the origin, by means of the Fuchs theory. By comparing, in section 5, these different families of fundamental solutions, this yields a set of functional relations which are detailed in section 6. Besides describing these general properties, we provide different examples in section 7 and appendix A, which will serve as guide lines in our next papers. In particular, for $m = 3$, we provide a family of isomonodromic deformations conditions.

In a second paper, we shall investigate the asymptotics of the solutions and Stokes multipliers with respect to the parameter $\underline{a}$, when $\|\underline{a}\| \to \infty$. Roughly speaking, this corresponds to the second part of Sibuya's work [35]. However, this work will be done in the framework

of the exact WKB analysis, thus taking advantage of the tools and ideas of the (quantum) resurgence theory, in the spirit of [40, 12].

A third paper will be devoted to applications to spectral analysis, especially for PT-symmetric operators $H = -\dfrac{d^2}{dx^2} + \dfrac{P_m(x,\underline{a})}{x^2}$, with a generalization of the result of Dorey et al [17] as an interesting by-product.

2 Solutions of $(\mathfrak{E}_m)$ in the neighbourhood of infinity: classical asymptotics

We begin in the framework of the usual Poincaré asymptotic analysis, (see, e.g., [16, 22, 30, 42]). We are interested in the question of the existence of solutions (formal or not) at infinity for equation $(\mathfrak{E}_m)$. The starting point of our study will be the main existence theorem 2.1, which can be seen as an adaptation of a classical theorem due to Sibuya ([35], p. 15). It asserts the existence and the unicity of a solution of $(\mathfrak{E}_m)$ defined by its asymptotic expansion at infinity.

2.1 The main existence theorem

In the sequel, it will be convenient to think of x as an element of the universal covering of $\mathbb{C}^\star$ with base point 1. Since this covering can be identified to $\mathbb{C}$ provided with the projection $t \mapsto x = e^t$, we shall associated to x its argument $\arg(x) \in \mathbb{R}$.

In what follows,

$$\frac{\sqrt{P_m(x,\underline{a})}}{x} = x^{\frac{m}{2}-1} + \sum_{k=1}^{N} b_{\frac{m}{2}-k}(\underline{a}) x^{\frac{m}{2}-k-1} + O(x^{\frac{m}{2}-N})$$

stands for the asymptotic expansion at infinity in x of $\dfrac{\sqrt{P_m(x,\underline{a})}}{x}$.

Theorem 2.1. *The differential equation $(\mathfrak{E}_m)$ admits a unique solution $\Phi_0(x,\underline{a})$ satisfying the following condition 1. :*

- *1. Φ_0 is an analytic function in x in the sector $\Sigma_0 = \{|x| > 0, \mid \arg(x) \mid < \frac{3\pi}{m}\}$ such that, in any strict sub-sector of Σ_0, Φ_0 admits an asymptotic expansion at infinity of the following form* [1]

$$T\Phi_0(x,\underline{a}) = x^{r(\underline{a})} e^{-S(x,\underline{a})} \phi_0(x,\underline{a}),$$

uniformly with respect to $\underline{a}$ in any compact set of $\mathbb{C}^m$, where:

$$- \textit{ i). if } m \textit{ is odd, } \begin{cases} S(x,\underline{a}) = \dfrac{2}{m} x^{\frac{m}{2}} + \displaystyle\sum_{k=1}^{\frac{m-1}{2}} \dfrac{b_{\frac{m}{2}-k}(\underline{a})}{\frac{m}{2}-k} x^{\frac{m}{2}-k} \in \mathbb{C}[\underline{a}][x^{\frac{1}{2}}] \\ \\ r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4} \\ \\ \phi_0 \in \mathbb{C}[\underline{a}][[x^{-\frac{1}{2}}]] \textit{ with constant term } 1. \end{cases}$$

[1]Throughout this theorem, $x^\alpha = \exp(\alpha \ln(x))$ with $\ln(x)$ real for $\arg(x) = 0$.

– *ii). if m is even,*
$$\begin{cases} S(x,\underline{a}) = \dfrac{2}{m}x^{\frac{m}{2}} + \displaystyle\sum_{k=1}^{\frac{m}{2}-1} \dfrac{b_{\frac{m}{2}-k}(\underline{a})}{\frac{m}{2}-k} x^{\frac{m}{2}-k} \in \mathbb{C}[\underline{a}][x] \\ \\ r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4} - b_0(\underline{a}) \\ \\ \phi_0 \in \mathbb{C}[\underline{a}][[x^{-1}]] \text{ with constant term } 1. \end{cases}$$

Moreover:

- *2. Φ_0 extends analytically in x to the universal covering of $\mathbb{C}^\star$, and is an entire function in $\underline{a}$.*
- *3. The derivative Φ_0' of Φ_0 with respect to x admits an asymptotic expansion at infinity of the form:*
$$T\left(\frac{d}{dx}\Phi_0(x,\underline{a})\right) = \frac{d}{dx}\left(T\Phi_0(x,\underline{a})\right) = x^{r(\underline{a})+\frac{m}{2}-1}e^{-S(x,\underline{a})}(-1+o(1))$$
when x tends to infinity in any strict sub-sector of Σ_0, uniformly with respect to $\underline{a}$.

Needless to say, the asymptotic expansion $T\Phi_0(x,\underline{a})$ of Φ_0 at infinity in Σ_0 can be computed algorithmically. For instance, for $m=3$ one gets (with Maple)

$$T\Phi_0(x,\underline{a}) = e^{-\frac{2}{3}x^{3/2}-a_1x^{1/2}}x^{-\frac{1}{4}}\times$$
$$\left(1+(a_2-\tfrac{1}{4}a_1^2)x^{-1/2}+(-\tfrac{1}{4}a_1^2a_2+\tfrac{1}{32}a_1^4-\tfrac{1}{4}a_1+\tfrac{1}{2}a_2^2)x^{-1}+O(x^{-3/2})\right),$$

while for $m=4$:

$$T\Phi_0(x,\underline{a}) = e^{(-\frac{1}{2}x^2-\frac{1}{2}a_1x)}\,x^{(-\frac{1}{2}-\frac{1}{2}a_2+\frac{1}{8}a_1^2)}\times$$
$$\left(1+(\tfrac{1}{16}a_1^3-\tfrac{1}{4}a_1a_2-\tfrac{1}{4}a_1+\tfrac{1}{2}a_3)x^{-1}+(\tfrac{5}{32}a_1^2-\tfrac{1}{16}a_2^2-\tfrac{1}{64}a_1^4a_2+\tfrac{1}{32}a_1^2a_2^2+\tfrac{5}{32}a_1^2a_2-\right.$$
$$\left.\tfrac{1}{8}a_1a_2a_3+\tfrac{1}{4}a_4-\tfrac{1}{4}a_2-\tfrac{9}{256}a_1^4-\tfrac{1}{4}a_1a_3-\tfrac{3}{16}+\tfrac{1}{512}a_1^6+\tfrac{1}{32}a_1^3a_3+\tfrac{1}{8}a_3^2)x^{-2}+O(x^{-3})\right).$$

We shall discuss the proof of theorem 2.1 in a moment (§3). Here, we would like to show how one can derive fundamental systems of solutions of $(\mathfrak{E}_m)$ from Φ_0 only. This is the subject of the following subsection.

2.2 Stokes-Sibuya coefficients

In the sequel, it will be convenient to introduce the following notations:

Notation 2.2. For all $\lambda \in \mathbb{C}$ and all $\underline{a} = (a_1, \cdots, a_m) \in \mathbb{C}^m$, we note
$$\lambda.\underline{a} := (\lambda a_1, \cdots, \lambda^m a_m).$$
We set:
$$\omega = e^{\pm\frac{2i\pi}{m}}$$
and we introduce:
$$\forall k \in \mathbb{Z},\quad \Phi_k(x,\underline{a}) = \Phi_0(\omega^k x, \omega^k.\underline{a}).$$
where Φ_0 is given by theorem 2.1.

We bring into play a quasi-homogeneity property of equation $(\mathfrak{E}_m)$. We note that equation $(\mathfrak{E}_m)$ is invariant under the transformation $(x, a) \mapsto (\omega x, \omega.\underline{a})$ so that, with the above notations, theorem 2.1 easily translates into the following lemma :

Lemma 2.3. *For any $k \in \mathbb{Z}$, Φ_k is a solution of $(\mathfrak{E}_m)$, and is entire in a. Its asymptotic expansion when x tends to infinity in the sector $\Sigma_k = \{|x| > 0, \mid arg(x) + k.arg(\omega) \mid < \frac{3\pi}{m}\}$, uniformly in $\underline{a}$ in any compact set of $\mathbb{C}^m$, is given by :*

$$T\Phi_k(x, \underline{a}) = T\Phi_0(\omega^k x, \omega^k.\underline{a})$$

where $T\Phi_0$ is the asymptotic expansion of Φ_0 in Σ_0 described in theorem 2.1.

We deduce the following corollary :

Corollary 2.4. *For every $k \in \mathbb{Z}$, the solution Φ_k is exponentially decreasing ("subdominant function" in [35], p. 19) in the sector $\Lambda_k = \{\mid arg(x) + k.arg(\omega) \mid < \frac{\pi}{m}\}$.*

We note that the sectors Λ_{k-1}, Λ_k and Λ_{k+1} are included in Σ_k and, by the previous lemma 2.3, each solution Φ_k has an exponential growth of order not greater than $\dfrac{m}{2}$ in the two sectors Λ_{k-1} and Λ_{k+1} adjacent to Λ_k. This allows to show the following lemma:

Lemma 2.5. *For every $k \in \mathbb{Z}$, $\{\Phi_k, \Phi_{k+1}\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$ and moreover,*

$$W(\Phi_k, \Phi_{k+1}) = 2(-1)^k \omega^{k(1-\frac{m}{2})+r(\omega^{k+1}.a)}$$

where W denotes the Wronskian, while r is given by theorem 2.1 .

Proof. By quasi-homogeneity of $P_m(x, \underline{a})$, we note that

$$S(\omega x, \omega.\underline{a}) = -S(x, \underline{a}).$$

Thus, by lemma 2.3, for $x \in \Sigma_k$,

$$T\Phi_k(x, \underline{a}) = \omega^{kr(\omega^k.\underline{a})} x^{r(\omega^k.\underline{a})} e^{(-1)^{k-1}S(x,\underline{a})}(1 + o(1)).$$

Using part 3. of Theorem 2.1 we have also, for $x \in \Sigma_k$,

$$T\Phi'_k(x, \underline{a}) = (-1)^{k-1} \omega^{kr(\omega^k.\underline{a})} x^{r(\omega^k.\underline{a})+\frac{m}{2}-1} e^{(-1)^{k-1}S(x,\underline{a})}(1 + o(1)).$$

Moreover, the coefficient b_0 of theorem 1 is a quasi-homogeneous polynomial in $\underline{a}$ of order $\frac{m}{2}$ so that :

$$r(\omega^k.\underline{a}) + r(\omega^{k+1}.\underline{a}) = 1 - \frac{m}{2}.$$

As a result, we get the equalities : for $x \in \Sigma_k \cap \Sigma_{k+1}$,

$$W(\Phi_k, \Phi_{k+1}) = \Phi_k \Phi'_{k+1} - \Phi'_k \Phi_{k+1}$$

$$= 2(-1)^k \omega^{k(r(\omega^k.\underline{a})+r(\omega^{k+1}.a))+r(\omega^{k+1}.\underline{a})} x^{r(\omega^k.\underline{a})+r(\omega^{k+1}.\underline{a})+\frac{m}{2}-1}(1 + o(1))$$

$$= 2(-1)^k \omega^{k(1-\frac{m}{2})+r(\omega^{k+1}.\underline{a})}(1 + o(1)).$$

The Wronskian $W(\Phi_k, \Phi_{k+1})$ being independent of x, this completes the proof. $\square$

Since each system $\{\Phi_k, \Phi_{k+1}\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$ we deduce, from the classical theory on linear differential equations, the existence of functions $C_k(\underline{a})$, $\widetilde{C}_k(\underline{a})$ depending only on the variable $\underline{a}$, such that :

$$\forall k \in \mathbb{Z}, \Phi_{k-1} = C_k(\underline{a})\Phi_k + \widetilde{C}_k(\underline{a})\Phi_{k+1}. \tag{1}$$

Definition 2.6. The functions $C_k(\underline{a})$ and $\widetilde{C}_k(\underline{a})$ defined by (1) are called the *Stokes-Sibuya coefficients* of Φ_{k-1} associated respectively with Φ_k and Φ_{k+1}. The matrices $\mathfrak{S}_k(\underline{a}) := \begin{pmatrix} C_k(\underline{a}) & \widetilde{C}_k(\underline{a}) \\ 1 & 0 \end{pmatrix}$ are called the *Stokes-Sibuya connection matrices.*

Differentiating the previous equalities (1) with respect to the variable x, we obtain the following formulas :

$$C_k(\underline{a}) = \frac{W(\Phi_{k-1}, \Phi_{k+1})}{W(\Phi_k, \Phi_{k+1})} \tag{2}$$

and

$$\widetilde{C}_k(\underline{a}) = \frac{W(\Phi_{k-1}, \Phi_k)}{W(\Phi_{k+1}, \Phi_k)}. \tag{3}$$

Using the fact that the Φ_k's are entire functions in $\underline{a}$, we deduce from (2), (3) and lemma 2.5 that the Stokes-Sibuya coefficients are entire functions in $\underline{a}$. Also, it follows from the very definition of the Φ_k's, from (3) and lemma 2.5 that $C_k(\underline{a}) = C_0(\omega^k.\underline{a})$, while $\widetilde{C}_k(\underline{a}) = \widetilde{C}_0(\omega^k.\underline{a}) = \omega^{m-2+2r(\omega^k.\underline{a})}$. In particular, since $\omega^m = e^{\pm 2i\pi}$, we get: for all $k \in \mathbb{Z}$, $C_k = C_{k \bmod m}$ and $\widetilde{C}_k = \widetilde{C}_{k \bmod m}$.

We summarize our results.

Theorem 2.7. *For all $k \in \mathbb{Z}$ we note*

$$\Phi_k(x, \underline{a}) = \Phi_0(\omega^k x, \omega^k.\underline{a}), \tag{4}$$

where Φ_0 is the solution of $(\mathfrak{E}_m)$ defined in theorem 2.1. Then, for every $k \in \mathbb{Z}$,

- *$\Phi_k(x, \underline{a})$ is analytic in x on the universal covering of $\mathbb{C}^\star$ and entire in $\underline{a}$.*
- *The system $\{\Phi_k, \Phi_{k+1}\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$.*
- *We have*

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a}) = \mathfrak{S}_k(\underline{a}) \begin{pmatrix} \Phi_k \\ \Phi_{k+1} \end{pmatrix}(x, \underline{a}), \tag{5}$$

  where the Stokes-Sibuya connection matrix $\mathfrak{S}_k(\underline{a})$ is invertible, and entire in $\underline{a}$. Moreover,

$$\mathfrak{S}_k(\underline{a}) = \mathfrak{S}_{k-1}(\omega.\underline{a}), \quad \mathfrak{S}_k(\underline{a}) = \mathfrak{S}_0(\omega^k.\underline{a}). \tag{6}$$

- *The Stokes-Sibuya coefficients $C_k(\underline{a})$ and $\widetilde{C}_k(\underline{a})$ associated respectively with Φ_k and Φ_{k+1} are entire functions in $\underline{a}$ and,*

$$\begin{cases} C_k(\underline{a}) = C_0(\omega^k.\underline{a}), \quad C_k = C_{k \bmod m} \\ \widetilde{C}_k(\underline{a}) = \widetilde{C}_0(\omega^k.\underline{a}) = \omega^{m-2+2r(\omega^k.\underline{a})}, \quad \widetilde{C}_k = \widetilde{C}_{k \bmod m} \end{cases}. \tag{7}$$

For any $k \in \mathbb{Z}$, the analytic continuation of $\{\Phi_{k-1}, \Phi_k\}$ constitutes a fundamental system of solutions of $(\mathfrak{E}_m)$ (by lemma 2.5, the Wronskian $W(\Phi_{k-1}, \Phi_k)$ does not vanish). In particular, there exists a unique invertible 2×2 matrix $\mathfrak{M}_k^\infty(\underline{a})$, entire in $\underline{a}$, such that $\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(\omega^m x, \underline{a}) = \mathfrak{M}_k^\infty(\underline{a}) \begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a})$.

Definition 2.8. The 2×2 matrices $\mathfrak{M}_k^\infty(\underline{a})$, $k \in \mathbb{Z}$, defined by

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(\omega^m x, \underline{a}) = \mathfrak{M}_k^\infty(\underline{a}) \begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a}), \tag{8}$$

are called the ∞-monodromy matrices.

From the very definition of the Φ_k's, we note that $\mathfrak{M}_k^\infty(\underline{a}) = \mathfrak{M}_0^\infty(\omega^k.\underline{a})$. Also,

$$\begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}(x, \underline{a}) = \mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) \begin{pmatrix} \Phi_{m-1} \\ \Phi_m \end{pmatrix}(x, \underline{a})$$

$$= \mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) \begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}(\omega^m x, \omega^m.\underline{a}).$$

Since $\begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}$ is entire in $\underline{a}$, we obtain

$$\begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}(x, \underline{a}) = \mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) \begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}(\omega^m x, \underline{a})$$

$$= \mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) \mathfrak{M}_0^\infty(\underline{a}) \begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}(x, \underline{a}),$$

and $\{\Phi_{-1}, \Phi_0\}$ being a fundamental system, this yields:

Theorem 2.9. *For every $k \in \mathbb{Z}$, the ∞-monodromy matrix $\mathfrak{M}_k^\infty(\underline{a})$ is invertible, entire in $\underline{a}$, and*

$$\mathfrak{M}_k^\infty(\underline{a}) = \mathfrak{M}_0^\infty(\omega^k.\underline{a}). \tag{9}$$

Furthermore, the Stokes-Sibuya connection matrices satisfy the functional relation:

$$\mathfrak{S}_0(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) = (\mathfrak{M}_0^\infty(\,\underline{a}))^{-1}. \tag{10}$$

Relation (10) generalized a functional relation due to Sibuya [35], p. 85. Unfortunately, as a rule, the ∞- monodromy matrix $\mathfrak{M}_0^\infty$ is difficult to compute. We return to this question in section 6.

3 Solutions of $(\mathfrak{E}_m)$ in the neighbourhood of infinity: resurgent point of view

Theorem 2.1 can be shown with the methods developed in Sibuya's book [35] and, in fact, theorem 2.1 is actually proved in [29, 1]. However, using the resurgent viewpoint, one can get a stronger result in a simpler way.

3.1 Basic notions in resurgence theory

Since the terminologies we shall use in this section are likely to be least familiar to the readers, we first introduce the necessary definitions, cf. [4, 5, 11, 18] for more details. We mention that our notations differ from those usually used by Ecalle.
As usual in this article, we identify an element of the universal covering of $\mathbb{C}^\star$ (with base point 1) by specifying its argument in $\mathbb{R}$.

Definition 3.1. A *sectorial neighbourhood of infinity of aperture* $I =]\alpha, \beta[\subset \mathbb{R}$ is an open set U of the universal covering of $\mathbb{C}^\star$ such that for any open interval $J \subset I$, there is $z \in U$ such that $zJ \subset U$, where

$$zJ := \{z + re^{i\theta},\ r > 0,\ \theta \in J\}.$$

Definition 3.2. If U is a sectorial neighbourhood of infinity of aperture I and if Ψ is holomorphic in U, Ψ is *of exponential growth of order 1 at infinity in* U if for any open interval $J \subset I$, there exist $\tau > 0$ and $C > 0$ such that

$$\forall z \in U \cap 0J,\ |\Psi(z)| \leq Ce^{\tau|z|}.$$

We now introduce the notion of minor. We do that only for a class of formal power series which will be used in this paper.

Definition 3.3. We consider the formal power series $\psi(z) = r + \displaystyle\sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$, where m is a positive integer. Then, r is the *residual coefficient* of ψ and $\widetilde{\psi}(\zeta) = \displaystyle\sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(\frac{n}{m}+1)} \zeta^{\frac{n}{m}} \in \mathbb{C}[[\zeta^{\frac{1}{m}}]]$ is the *minor* of ψ.

In other words, the minor of the formal power series ψ is nothing but its Borel transform when forgetting its residual coefficient.

This allows to define the Borel-summability for such formal series.

Definition 3.4. The formal power series $\psi(z) = r + \displaystyle\sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ is *Borel-resummable in the direction (of argument)* $\alpha \in \mathbb{R}$ if:

1. its minor $\widetilde{\psi}(\zeta)$ defines a (ramified) analytic function at the origin[2],
2. there exists an open sector $0I$ with I an open neighbourhood of α such that $\widetilde{\psi}(\zeta)$ can be analytically extended in $0I$ and is of exponential growth of order 1 at infinity in $0I$.

The *Borel-sum* $\mathrm{s}_\alpha\psi(z)$ with respect to z in the direction $\alpha \in \mathbb{R}$ of the formal series ψ is defined by

$$\mathrm{s}_\alpha\psi(z) := r + \int_0^{\infty e^{i\alpha}} e^{-z\zeta}\widetilde{\psi}(\zeta)d\zeta.$$

[2]For simplicity, we keep the same notation $\widetilde{\psi}(\zeta)$ for the series and its sum, and ζ should be seen as an element of the universal covering of $\mathbb{C}^\star$.

In definition 3.4, when one drops the growth condition at infinity (condition 2), then ψ is said to be *Borel presummable* in the direction α, the summation operator s_α being replaced by the *presummation* operator which we do not define here, see, e.g., [11].

A Borel sum has the following main properties:

Proposition 3.5. *If* $\psi(z) = r + \sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ *is Borel-resummable in the direction* $\alpha \in \mathbb{R}$*, then:*

- *its Borel sum* $\mathrm{s}_\alpha\psi(z)$ *is holomorphic in a sectorial neighbourhood of infinity* U *of aperture* $I =]-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha[$.
- $\mathrm{s}_\alpha\psi(z)$ *is asymptotic to* $\psi(z)$ *at infinity in* U*. More precisely, for any strict subinterval* J *of* I*, there is* $C > 0$ *such that, for all* $n \geq 1$*, for all* $z \in U \cap 0J$*,* $|\mathrm{s}_\alpha\psi(z) - r - \sum_{k=0}^{n-1} \frac{\alpha_k}{z^{\frac{k}{m}+1}}| \leq C^n \Gamma(\frac{n}{m} + 1)|z|^{-\frac{n}{m}-1}$.
- $\frac{d}{dz}\left(\mathrm{s}_\alpha\psi(z)\right) = \mathrm{s}_\alpha\left(\frac{d\psi}{dz}(z)\right)$.
- *If two formal power series* $\psi(z)$*,* $\phi(z) \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ *are Borel-resummable in the direction* $\alpha \in \mathbb{R}$*, then* $\mathrm{s}_\alpha\left(\psi.\phi\right)(z) = \mathrm{s}_\alpha(\psi)(z).\mathrm{s}_\alpha(\phi)(z)$.

Definition 3.6. A formal power series $\psi(z) = r + \sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ is *resurgent* if its minor $\widetilde{\psi}(\zeta)$ defines a (ramified) analytic function at the origin and is *endlessly continuable*, i.e., for every $L > 0$ there is a finite subset $\Omega_L \subset \mathbb{C}$ such that $\widetilde{\psi}$ can be analytically continued along every path λ of length $< L$ which avoids Ω_L.

This definition[3] can be extended to an algebra of *resurgent formal functions* which we do not precise here.

Proposition 3.7. *If two formal power series* $\psi(z) = r + \sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ *and* $\phi(z) = s + \sum_{n=0}^{\infty} \frac{\beta_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$ *are resurgent, then the product* $\psi.\phi(z)$ *is also resurgent: the minor of* $\psi.\phi(z)$*, which is given by*

$$\begin{aligned} \widetilde{\psi.\phi}(\zeta) &= r.\widetilde{\phi}(\zeta) + s.\widetilde{\psi}(\zeta) + \widetilde{\psi} * \widetilde{\phi}(\zeta), \\ \widetilde{\psi} * \widetilde{\phi}(\zeta) &= \int_0^\zeta \widetilde{\psi}(\eta)\widetilde{\phi}(\zeta - \eta)d\eta \text{ (the convolution product),} \end{aligned} \tag{11}$$

is endlessly continuable.

[3]Note that Ecalle proposes a more general definition.

For a resurgent formal power series, it may happen that we no longer can define its Borel (pre)sum in a given direction $\alpha \in \mathbb{R}$ because the minor can have singularities along this direction: this is the essence of the *Stokes phenomenon*.

Definition 3.8. We consider a resurgent formal power series $\psi(z) = r + \sum_{n=0}^{\infty} \frac{\alpha_n}{z^{\frac{n}{m}+1}} \in \mathbb{C}[[z^{-\frac{1}{m}}]]$. Let $\alpha \in \mathbb{R}$ be a singular direction for the minor $\widetilde{\psi}(\zeta)$.
Hypothesis: we assume that there is $\epsilon > 0$ such that $\widetilde{\psi}(\zeta)$ can be analytically extended in the open sector $0]\alpha, \alpha+\epsilon[$ (resp. $0]\alpha-\epsilon, \alpha[$ with an exponential growth of order 1 at infinity. We assume also that this exponential growth at infinity extends up to a path $[0, \infty e^{i\alpha}+[$ (resp. $[0, \infty e^{i\alpha}-[$) which circumvents the singularities to the left (resp. right) along the direction α, see figure 1.
Then ψ is *right* (resp. *left*) *Borel-resummable* in the direction α, its right (resp. left) Borel sum $\mathrm{s}_{\alpha+}\psi$ (resp. $\mathrm{s}_{\alpha-}\psi$) being defined by

$$\mathrm{s}_{\alpha\pm}\psi(z) := r + \int_0^{\infty e^{i\alpha}\pm} e^{-z\zeta}\widetilde{\psi}(\zeta)d\zeta,$$

for z in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{2}-\alpha, \frac{\pi}{2}-\alpha[$.

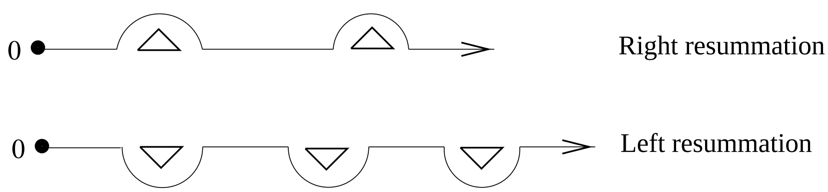

Figure 1: The integration path for *right* (resp. *left*) resummation (for $\alpha = 0$).

In definition 3.8, it is possible to drop the Hypothesis, replacing right (resp. left) Borel sum by *right* (resp. *left*) *Borel presum*, see, e.g., [11]. In other words, every resurgent formal function is always right and left Borel-presummable (in any direction).

We note that proposition 3.16 is still valid for right and left Borel sum. Moreover, when ψ is Borel (pre)summable in the direction $\alpha \in \mathbb{R}$, then

$$\mathrm{s}_{\alpha}\psi(z) = \mathrm{s}_{\alpha+}\psi(z) = \mathrm{s}_{\alpha-}\psi(z).$$

In order to be able to compare right and left (pre)summation, one has to enlarge the set of resurgent formal functions to the set of resurgent symbols.

Definition 3.9. A *resurgent symbol*[4] in the direction α is a formal sum

$$\varphi(z) = \sum_{\omega \in \Omega} \psi_\omega \, e^{-z\omega}$$

where each $\psi_\omega(z)$ is a resurgent formal function and Ω, the *singular support of* φ, is a discrete subset of $[0, \infty e^{i\alpha}[$.
The sum and product of two resurgent symbols are defined in an obvious fashion, so that resurgent symbols in the direction α make up an algebra which we denote by $\mathcal{R}_\alpha$.

[4] or *resurgent transseries*.

The *right* (resp. *left*) (pre)summation operations can be extended to resurgent symbols in a way so that

$$\mathrm{s}_{\alpha+}\,\varphi = \sum_{\omega\in\Omega} (\mathrm{s}_{\alpha+}\,\psi_\omega)\,e^{-z\omega} \quad \text{resp.} \quad \mathrm{s}_{\alpha-}\,\varphi = \sum_{\omega\in\Omega} (\mathrm{s}_{\alpha-}\,\psi_\omega)\,e^{-z\omega}.$$

The construction (which we do not explain here) makes the operations $\mathrm{s}_{\alpha+}$ and $\mathrm{s}_{\alpha-}$ *isomorphisms of algebras* and, moreover, $\mathrm{s}_{\alpha+}(\mathcal{R}_\alpha) = \mathrm{s}_{\alpha-}(\mathcal{R}_\alpha)$. This key-result (due to Ecalle) allows to define the so-called Stokes automorphism, which analyzes the Stokes phenomenon by comparing right and left Borel-(pre)summations:

Definition 3.10. The *Stokes automorphism in the direction* α is defined by:

$$\mathfrak{S}_\alpha := \mathrm{s}_{\alpha+}^{-1} \circ \mathrm{s}_{\alpha-} \,:\, \mathcal{R}_\alpha \to \mathcal{R}_\alpha.$$

Remark: the action of the Stokes automorphism in a given direction can be understood in terms of deformation of the contour of integration in a Laplace integral, see Figure 2.

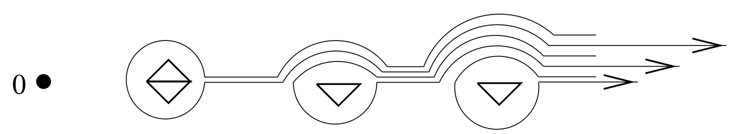

Figure 2: The difference between right and left summations (Fig. 1) is described as a Laplace integral along a sum of contours.

It follows from the definitions that the Stokes automorphism in the direction α acts trivially on exponentials $e^{-z\omega}$, and that its action on a formal resurgent series ψ reads:

$$\mathfrak{S}_\alpha \psi = \psi + \sum_{\omega} \psi_\omega e^{-z\omega},$$

where the sum runs over those singular points of the minor $\widetilde{\psi}$ which have to be avoided when considering left (pre)summation. The Stokes automorphism $\mathfrak{S}_\alpha$ reads as

$$\mathfrak{S}_\alpha = \mathbb{1} + {}^{+}\mathbb{S}_\alpha$$

where the operator ${}^{+}\mathbb{S}_\alpha$ commutes with multiplication by exponentials, and transforms formal resurgent series into “exponentially small resurgent symbols”. This implies that the operator

$$\dot{\underline{\Delta}}_\alpha := \ln \mathfrak{S}_\alpha = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} {}^{+}\mathbb{S}_\alpha^n$$

is well defined on $\mathcal{R}_\alpha$. Since $\mathfrak{S}_\alpha$ is an automorphism of $\mathcal{R}_\alpha$, $\dot{\underline{\Delta}}_\alpha$ is a derivation of $\mathcal{R}_\alpha$.

Definition 3.11. $\dot{\underline{\Delta}}_\alpha$ is called the *alien derivation in the direction* α.

We note that since $\mathrm{s}_{\alpha+}$ and $\mathrm{s}_{\alpha-}$ commutes with $\frac{d}{dz}$, $\mathfrak{S}_\alpha$ and therefore $\dot{\underline{\Delta}}_\alpha$ commutes with $\frac{d}{dz}$.

Proposition 3.12. $\dot{\underline{\Delta}}_\alpha$ *commutes with* $\frac{d}{dz}$.

The alien derivation $\dot{\underline{\Delta}}_\alpha$ commutes with multiplication by exponentials, and its action on a formal resurgent series ψ has the following explicit form:

$$\dot{\underline{\Delta}}_\alpha \psi = \sum_{\omega \in \Omega(\widetilde{\psi})} e^{-z\omega} \, \Delta_\omega \psi,$$

where $\Omega(\widetilde{\psi})$ is a discrete subset of $[0, \infty e^{i\alpha}[$, the set of so-called *glimpsed singularities* of $\widetilde{\psi}$, i.e., the set of singularities to be circumvented when analytically continuing $\widetilde{\psi}$ in the direction α. Since $\dot{\underline{\Delta}}_\alpha$ is a derivation, each Δ_ω is also a derivation.

Definition 3.13. Δ_ω is called the *alien derivation at* ω.

Definition 3.14. A formal resurgent function ψ is a *constant of resurgence* if for any ω, $\Delta_\omega \psi = 0$.

For instance, if ψ is a convergent power series, then ψ is a constant of resurgence.

Instead of working with the Δ_ω 's, it is sometimes convenient to work with the so-called *pointed alien derivations*

$$\dot{\Delta}_\omega \, := \, e^{-z\omega} \, \Delta_\omega.$$

They have the advantage of commuting with $\dfrac{d}{dz}$ (this is a consequence of proposition 3.12).

Proposition 3.15. *The pointed alien derivations* $\dot{\Delta}_\omega$ *commute with* $\frac{d}{dz}$.

3.2 Resurgence of solutions of $(\mathfrak{E}_m)$

We now return to the proof of theorem 2.1. The main idea will be to consider the asymptotic expansion $T\Phi_0$ of theorem 2.1 as a formal solution of equation $(\mathfrak{E}_m)$, and to show that Φ_0 can be constructed from $T\Phi_0$ by Borel resummation with respect to an appropriate variable $z = z(x, \underline{a})$ (or $\widetilde{z}$, depending on the parity of m) which will be defined later, uniformly in $\underline{a}$ for $\underline{a}$ in any given compact set.

We start with a kind of preparation theorem, so as to transform equation $(\mathfrak{E}_m)$ into a normal form. This is based on the Green-Liouville transformation, but we shall have to dissociate the m odd case from the m even case for technical reasons.

In what follows, $\underline{a}$ is assumed to belong to an arbitrary given compact set $K \subset \mathbb{C}^m$.

3.2.1 The m odd case

We consider the Green-Liouville transformation :

$$\begin{cases} z = z(x, \underline{a}) = \displaystyle\int^x \frac{\sqrt{P_m(t, \underline{a})}}{t} dt \\[2ex] \Psi(z, \underline{a}) = \dfrac{P_m(x, \underline{a})^{\frac{1}{4}}}{\sqrt{x}} \Phi(x, \underline{a}) \end{cases} \tag{12}$$

where the Laurent-Puiseux series expansion in x of $z(x,\underline{a})$,

$$z(x,\underline{a})=\frac{2}{m}x^{\frac{m}{2}}+\sum_{k=1}^{\frac{m-1}{2}}\frac{b_{\frac{m}{2}-k}(\underline{a})}{\frac{m}{2}-k}x^{\frac{m}{2}-k}+O(x^{-\frac{1}{2}})\in x^{\frac{m}{2}}\mathbb{C}[\underline{a}]\{x^{-1}\},$$

coincides with the map $x\mapsto S(x,\underline{a})$ defined in theorem 2.1 modulo an analytic (multivalued) function vanishing at infinity. We note that, if $\begin{array}{c}X\\ \downarrow\pi\\ \mathbb{C}\end{array}$ (*resp.* $\begin{array}{c}Z\\ \downarrow\widetilde{\pi}\\ \mathbb{C}\end{array}$) is the (ramified) Riemann surface of $x^{1/2}$ (*resp.* $z^{1/m}$), then there exists a compact set B (depending on K) (*resp.* $\widetilde{B}$) such that the map $(x,\underline{a})\in\pi^{-1}(\mathbb{C}\backslash B)\times K\mapsto(z(x,\underline{a}),\underline{a})\in\widetilde{\pi}^{-1}(\mathbb{C}\backslash\widetilde{B})\times K$ is bi-holomorphic.

Remark here that by quasi-homogeneity of $P_m(x,\underline{a})$,

$$z(\omega x,\omega.\underline{a})=\omega^{\frac{m}{2}}z(x,\underline{a}). \tag{13}$$

The transformation (12) converts $(\mathfrak{E}_m)$ into the following equation,

$$-\frac{d^2}{dz^2}\Psi+(1-F(z,\underline{a}))\Psi=0, \tag{14}$$

which is our prepared normal form.

It is straightforward to see that studying $(\mathfrak{E}_m)$ at infinity in the variable x amounts to studying (14) at infinity in the variable z. The inverse map $x:(z,\underline{a})\mapsto x(z,\underline{a})$ can be identified with its Laurent-Puiseux series expansion

$$x(z,\underline{a})=\left(\frac{m}{2}z\right)^{\frac{2}{m}}+O(1)\in z^{\frac{2}{m}}\mathbb{C}[\underline{a}]\{z^{-\frac{2}{m}}\}, \tag{15}$$

and from (13),

$$x(\omega^{\frac{m}{2}}z,\omega.\underline{a})=\omega x(z,\underline{a}). \tag{16}$$

In (14), $F(z,\underline{a})$ is defined by:

$$F(z,\underline{a})=-\frac{xP_m'(x,\underline{a})+P_m(x,\underline{a})}{4P_m(x,\underline{a})^2}-x^2\Big(\frac{1}{4}\frac{P_m''(x,\underline{a})}{P_m(x,\underline{a})^2}-\frac{5}{16}\frac{(P_m'(x,\underline{a}))^2}{P_m(x,\underline{a})^3}\Big)\,|_{x\,=\,x(z,\underline{a})}. \tag{17}$$

One infers from (15) that

$$F(z,\underline{a})=\frac{m^2-4}{4m^2z^2}(1+O(z^{-\frac{2}{m}}))\in\frac{1}{z^2}\mathbb{C}[\underline{a}]\{z^{-\frac{2}{m}}\} \tag{18}$$

is an analytic function at infinity in $z^{-1/m}$, uniformly in $\underline{a}\in K$. It is easy to show the existence of a unique formal solution $\Psi_+(z,\underline{a})$ of (14) satisfying

$$\Psi_+(z,\underline{a})=e^{-z}\psi_+(z,\underline{a}), \tag{19}$$

where

$$\psi_+(z,\underline{a})=1+\sum_{n=0}^{\infty}\frac{\alpha_n(\underline{a})}{z^{\frac{n}{m}+1}}\in\mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]] \tag{20}$$

with 1 for its residual coefficient. Moreover the formal power series expansion $\psi_+(z,\underline{a})$ satisfies the following ordinary differential equation:

$$-\frac{d^2}{dz^2}\psi_+ + 2\frac{d}{dz}\psi_+ - F(z,\underline{a})\psi_+ = 0. \tag{21}$$

In addition, from the quasi-homogeneity property of P_m, from (16) et (17), one sees that

$$F(\omega^{\frac{m}{2}}z,\omega.\underline{a}) = F(z,\underline{a}). \tag{22}$$

Defining

$$\psi_-(z,\underline{a}) = \psi_+(\omega^{\frac{m}{2}}z,\omega.\underline{a}), \tag{23}$$

one deduces the existence of a unique formal solution $\Psi_-(z,\underline{a})$ of (14) such that $\Psi_-(z,\underline{a}) = e^z\psi_-(z,\underline{a})$ with $\psi_- \in \mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]$ with a residual coefficient equal to 1. Note that Ψ_+,Ψ_- are linearly independent, so that (Ψ_+,Ψ_-) provides a fundamental system of formal solutions for the linear second order equation (14).

The formal series $\psi_\pm$ enjoys the following properties:

Proposition 3.16. *The formal power series expansion* $\psi_+(z,\underline{a}) = 1+\sum_{n=0}^{\infty}\frac{\alpha_n(\underline{a})}{z^{\frac{n}{m}+1}} \in \mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]$ *(resp.* $\psi_-(z,\underline{a}) = \psi_+(\omega^{\frac{m}{2}}z,\omega.\underline{a})$*) is Borel-resummable with respect to* z*, uniformly in* $\underline{a}$ *for* $\underline{a}$ *in any compact set of* $\mathbb{C}^m$*, for every direction of summation except those of argument* π mod (2π) *(resp.* 0 mod (2π)*).*

Proof. We have to analyze the analytic properties of the minor

$$\widetilde{\psi}_+(\zeta,\underline{a}) = \sum_{n=0}^{\infty}\frac{\alpha_n(\underline{a})}{\Gamma(\frac{n}{m}+1)}\zeta^{\frac{n}{m}} \in \mathbb{C}[\underline{a}][[\zeta^{\frac{1}{m}}]]. \tag{24}$$

of $\psi_+(z,\underline{a})$. To proceed, we go back to equation (21). Instead of considering this differential equation, we shall rather introduce its deformation,

$$-\frac{d^2}{dz^2}\psi + 2\frac{d}{dz}\psi - F(z,\underline{a}) + \varepsilon F(z,\underline{a})(1-\psi) = 0, \tag{25}$$

where ε can be thought of as a parameter of perturbation. The introduction of this parameter will help us to rewrite ψ_+ and its minor $\widetilde{\psi}_+$ into an analyzable form, since (25) reduces to (21) when $\varepsilon = 1$. We now look for a formal solution of (25) in the form of a normalized series expansion with respect to ε:

$$\psi(z,\underline{a},\varepsilon) = 1 + \sum_{n=0}^{\infty}\psi_n(z,\underline{a})\varepsilon^n, \qquad \psi_n \in \frac{1}{z}\mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]. \tag{26}$$

Plugging (26) into (25) and identifying the powers of ε, one gets:

$$\begin{cases} -\dfrac{d^2}{dz^2}\psi_0 + 2\dfrac{d}{dz}\psi_0 = F(z,\underline{a}) \\ \\ -\dfrac{d^2}{dz^2}\psi_{n+1} + 2\dfrac{d}{dz}\psi_{n+1} = F(z,\underline{a})\psi_n,\ n \geq 0. \end{cases} \tag{27}$$

This translates into the fact that the minors $\widetilde{\psi}_n(\zeta,\underline{a})$ of the $\psi_n(z,\underline{a})$ have to satisfy the following convolution equations,

$$\left\{\begin{array}{l} -\zeta(2+\zeta)\widetilde{\psi}_0 = \widetilde{F} \\ \\ -\zeta(\zeta+2)\widetilde{\psi}_{n+1} = \widetilde{\psi}_n * \widetilde{F},\ n \geq 0, \end{array}\right. \tag{28}$$

where $\widetilde{F}(\zeta,\underline{a})$ is the minor of $F(z,\underline{a})$, while $*$ stands for the convolution product (cf. (11)). We have now to analyze the analytic properties of

$$\widetilde{\psi}(\zeta,\underline{a},\varepsilon) = \sum_{n=0}^{\infty} \widetilde{\psi}_n(\zeta,\underline{a})\varepsilon^n. \tag{29}$$

The key-point of the analysis will come from the properties of F. Writing

$$F(z,\underline{a}) = \sum_{n=0}^{\infty} \frac{f_n(\underline{a})}{z^{\frac{2n}{m}+2}} \in \frac{1}{z^2}\mathbb{C}[\underline{a}]\{z^{-\frac{2}{m}}\}, \tag{30}$$

we know that

$$G(z) = \sum_{n=0}^{\infty} \frac{g_n}{z^{\frac{2n}{m}+2}} \quad \text{with} \quad g_n = \sup_{\underline{a}\in K} |f_n(\underline{a})|, \tag{31}$$

is an analytic function at infinity in $z^{-1/m}$. Therefore, its minor

$$\widetilde{G}(\zeta) = \sum_{n=0}^{\infty} \frac{g_n}{\Gamma(\frac{2n}{m}+2)}\zeta^{\frac{2n}{m}+1} \in \zeta\mathbb{C}\{\zeta^{\frac{2}{m}}\} \tag{32}$$

is an entire function in $\zeta^{1/m}$ (with an exponential growth at infinity of order at most 1). Thus, if $\mathfrak{C}_m$ denotes the Riemann surface of $\zeta^{1/m}$, then

$$\widetilde{F}(\zeta,\underline{a}) = \sum_{n=0}^{\infty} \frac{f_n(\underline{a})}{\Gamma(\frac{2n}{m}+2)}\zeta^{\frac{2n}{m}+1} \in \zeta\mathbb{C}[\underline{a}]\{\zeta^{\frac{2}{m}}\}$$

is a holomorphic function in $(\zeta,\underline{a}) \in \mathfrak{C}_m \times K$ such that

$$\forall(\zeta,\underline{a}) \in \mathfrak{C}_m \times K,\ |\widetilde{F}(\zeta,\underline{a})| \leq G(|\zeta|). \tag{33}$$

Using the fact that $\widetilde{F}$ is a holomorphic function in $(\zeta,\underline{a}) \in \mathfrak{C}_m \times K$ such that $F(\zeta,\underline{a}) = O(\zeta)$ uniformly in $\underline{a} \in K$, and from the properties of the convolution product, one easily deduces from (28) that each $\widetilde{\psi}_n$ belongs to the space $\mathbb{C}[\underline{a}]\{\zeta^{\frac{1}{m}}\}$ and extends analytically to $\mathbb{C}\widetilde{\backslash\{0,-2\}} \times K$, where $\mathbb{C}\widetilde{\backslash\{0,-2\}}$ is the universal covering of $\mathbb{C}\backslash\{0,-2\}$.
For $\rho > 0$, we now define the star-shape domain

$$\Omega_m(\rho) = \{\zeta \in \mathfrak{C}_m,\ |\dot{\zeta}+2| > \rho,\ [0,\zeta] \in \Omega_m(\rho)\} \subset \mathfrak{C}_m \tag{34}$$

where $\dot{\zeta}$ is the projection of ζ by the natural mapping $\mathfrak{C}_m \to \mathbb{C}$. We also introduce the sequence of analytic functions $h_n(\zeta)$ defined for $\zeta \in \mathfrak{C}_m$ by:

$$\left\{\begin{array}{l} \zeta\rho\widetilde{h}_0 = \widetilde{G} \\ \\ \zeta\rho\widetilde{h}_{n+1} = \widetilde{h}_n * \widetilde{G},\ n \geq 0 \end{array}\right. . \tag{35}$$

Comparing (35) with (28), and using (33), one gets

$$\forall (\zeta, \underline{a}) \in \Omega_m(\rho) \times K,\ \forall n \in \mathbb{N},\ |\widetilde{\psi}_n(\zeta, \underline{a})| \leq \widetilde{h}_n(|\zeta|). \tag{36}$$

This can be shown by an easy recursion. We just detail here the $n = 0$ and $n = 1$ cases. For all $(\zeta, \underline{a}) \in (\Omega_m(\rho)\backslash\{0\}) \times K$ we first have:

$$|\widetilde{\psi}_0(\zeta, \underline{a})| = \frac{|\widetilde{F}(\zeta, \underline{a})|}{|\zeta||\zeta + 2|} \leq \frac{\widetilde{G}(|\zeta|)}{|\zeta|\rho} = \widetilde{h}_0(|\zeta|),$$

and this inequality extends to $\zeta = 0$ by continuity. This proves (36) for $n = 0$.
We thus deduce that, for all $(\zeta, \underline{a}) \in (\Omega_m(\rho)\backslash\{0\}) \times K$:

$$|\widetilde{\psi}_1(\zeta, \underline{a})| = \frac{|\widetilde{F} * \widetilde{\psi}_0(\zeta, \underline{a})|}{|\zeta||\zeta + 2|} \leq \frac{|\int_0^\zeta \widetilde{F}(\eta, \underline{a})\widetilde{\psi}_0(\zeta - \eta, \underline{a})d\eta|}{|\zeta|\rho}.$$

Writing $\zeta = |\zeta|e^{i\theta}$ and making the change of variable $\eta = te^{i\theta}$, we get:

$$\left|\int_0^\zeta \widetilde{F}(\eta, \underline{a})\widetilde{\psi}_0(\zeta - \eta, \underline{a})d\eta\right| = \left|\int_0^{|\zeta|} \widetilde{F}(te^{i\theta}, \underline{a})\widetilde{\psi}_0((|\zeta| - t)e^{i\theta}, \underline{a})dt\right|$$

$$\leq \int_0^{|\zeta|} |\widetilde{F}(te^{i\theta}, \underline{a})|.|\widetilde{\psi}_0((|\zeta| - t)e^{i\theta}, \underline{a})|dt \leq \int_0^{|\zeta|} \widetilde{G}(t)\widetilde{h}_0(|\zeta| - t)dt = \widetilde{G} * \widetilde{h}_0(|\zeta|).$$

Therefore, for all $(\zeta, \underline{a}) \in (\Omega_m(\rho)\backslash\{0\}) \times K$,

$$|\widetilde{\psi}_1(\zeta, \underline{a})| \leq \frac{\widetilde{G} * \widetilde{h}_0(|\zeta|)}{|\zeta|\rho} = \widetilde{h}_1(|\zeta|).$$

This gives (36) for $n = 1$ by an argument of continuity.
Now, $\widetilde{h}(\zeta, \varepsilon) = \sum_{n=0}^{\infty} \widetilde{h}_n(\zeta)\varepsilon^n$ is nothing but the minor of the series expansion $h(z, \varepsilon) = \sum_{n=0}^{\infty} h_n(z)\varepsilon^n$, where the h_n's are defined recursively by:

$$\begin{cases} -\rho \dfrac{d}{dz} h_0 = G \\ \\ -\rho \dfrac{d}{dz} h_{n+1} = h_n G,\ \ n \geq 0. \end{cases} \tag{37}$$

This means that h satisfies the following ordinary differential equation:

$$-\rho \frac{d}{dz} h = \varepsilon G(z) h + G(z). \tag{38}$$

From (31), we see that G is integrable at infinity, so that the function

$$(z, \varepsilon) \mapsto \frac{e^{-\frac{\varepsilon}{\rho}\int_{+\infty}^z G(z')dz'} - 1}{\varepsilon} \tag{39}$$

is a solution of equation (38) which is holomorphic for z in a neighbourhood of infinity of $\mathfrak{C}_m$ and $\varepsilon \in D(0, R)$, $R > 1$. Moreover, its Taylor series expansion at $\varepsilon = 0$ is exactly

$$h(z,\varepsilon)=\sum_{n=0}^{\infty} h_n(z)\varepsilon^n.$$

In return, this proves that $\widetilde{h}(\zeta,\varepsilon)$ defines a holomorphic function in $(\zeta,\varepsilon)\in\mathfrak{C}_m\times D(0,R)$, with an exponential growth of order not greater than 1 at infinity in ζ, uniformly in $\varepsilon\in D(0,R)$: there exist $A,B\in]0,+\infty[$ such that

$$\forall(\zeta,\varepsilon)\in\mathfrak{C}_m\times D(0,R),\ |h(\zeta,\varepsilon)|\leq Ae^{B|\zeta|}.$$

This last result, together with (36), shows that the series expansion $\widetilde{\psi}_n(\zeta,\underline{a},\varepsilon)$ converges uniformly for ζ in every compact set of $\Omega_m(\rho)$, $\underline{a}\in K$ and $\varepsilon\in D(0,R)$, and moreover,

$$\forall(\zeta,\underline{a},\varepsilon)\in\Omega_m(\rho)\times K\times D(0,R),\ |\widetilde{\psi}(\zeta,\underline{a},\varepsilon)|\leq\widetilde{h}(|\zeta|,|\varepsilon|)\leq Ae^{B|\zeta|}.$$

Putting $\varepsilon=1$, we deduce the same result for $\widetilde{\psi}_+(\zeta,\underline{a})$: holomorphy in $\Omega_m(\rho)\times K$, exponential growth of order not greater than 1 at infinity in ζ, uniformly in $\underline{a}\in K$.
Since $\rho>0$ can be chosen arbitrarily small, we have shown that, except for the directions of argument $\alpha=\pi \mod (2\pi)$, there is no singular point on the half line $\arg\zeta=\alpha$ and, $\widetilde{\psi}_+(\zeta,\underline{a})$ having an exponential growth of order not greater than 1 at infinity in ζ, uniformly in $\underline{a}\in K$, we deduce that $\psi_+(z,\underline{a})$ is Borel-resummable with respect to z, uniformly in $\underline{a}$ for $\underline{a}$ in any compact set of $\mathbb{C}^m$, for every direction of summation except those of argument $\pi \mod (2\pi)$.

Thanks to (23), an analogous result can be obtained $\psi_-(z,\underline{a})$. This yields proposition 3.16. □

Proposition 3.16 is enough to prove theorem 2.1. Let us define $\phi_0(x,\underline{a})\in\mathbb{C}[\underline{a}][[x^{-\frac{1}{2}}]]$ by the following formula :

$$x^{r(\underline{a})}e^{-S(x,\underline{a})}\phi_0(x,\underline{a})=\frac{\sqrt{x}}{P_m(x,\underline{a})^{\frac{1}{4}}}e^{-z}\psi_+(z,\underline{a})\,|_{z\,=\,z(x,\underline{a})}.$$

Due to the very definition of ψ_+, the left-hand side of this equality is a formal solution of equation $(\mathfrak{E}_m)$.
We know from proposition 3.16 that $\psi_+(z,\underline{a})$ is Borel-resummable for the direction of argument 0. For $\underline{a}$ in any given compact set K of $\mathbb{C}^m$, this allows us to define the function

$$\Phi_0(x,\underline{a})=\frac{\sqrt{x}}{P_m(x,\underline{a})^{\frac{1}{4}}}e^{-z}\,\mathrm{s}_0\psi_+(z,\underline{a})\,|_{z\,=\,z(x,\underline{a})},$$

which is an analytic solution of $(\mathfrak{E}_m)$ for z in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{2},\frac{\pi}{2}[$ and $\underline{a}$ in K. Note that the size of the sectorial neighbourhood may depend on K.
By the inverse map $z\leftrightarrow x$ (given by (15)), this corresponds to a x-sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{m},\frac{\pi}{m}[$. From proposition 3.16 again, Φ_0 can be analytically extended by varying the direction of summation on $]-\pi,\pi[$. This shows that Φ_0 is holomorphic in a x-sectorial neighbourhood of infinity Σ_0' of aperture $]-\frac{3\pi}{m},\frac{3\pi}{m}[$ and, by construction, Φ_0 is asymptotic to $x^{r(\underline{a})}e^{-S(x,\underline{a})}\phi_0(x,\underline{a})$ at infinity in Σ_0', uniformly in $\underline{a}\in K$.
The uniqueness of Φ_0 follows from the Watson theorem [27].
Also, since for any strict sub-sector Σ of Σ_0 the set $\Sigma\backslash\Sigma\cap\Sigma_0'$ is bounded, all we have to do now to get part (1) of theorem 2.1 is to show that Φ_0 extends analytically in $x\in\Sigma_0$. This is a consequence of the Cauchy-Kovalevskaya theorem: take a point x_0 in Σ_0' and consider the datum $(\Phi_0(x_0,\underline{a}),\Phi_0'(x_0,\underline{a}))$. Then Φ_0 is uniquely defined by this Cauchy datum, which

is holomorphic in $\underline{a} \in K$. Since the linear differential equation $(\mathfrak{E}_m)$ is holomorphic in $(x,\underline{a}) \in \mathbb{C}^\star \times \mathbb{C}$, we conclude that Φ_0 extends analytically to $\widetilde{\mathbb{C}^\star} \times K$, where $\widetilde{\mathbb{C}^\star}$ stands for the universal covering of $\mathbb{C}^\star$. We end by noticing that K can be chosen arbitrarily. This also shows part (2) of theorem 2.1.
Part (3) of theorem 2.1 follows from the fact that the Borel resummation with respect to z commutes with the derivative $\dfrac{d}{dz}$.

Note that besides proving theorem 2.1, we have obtained the following interesting result:

Proposition 3.17. *When m is odd, the analytic function Φ_0 of theorem 2.1 is given by*

$$\Phi_0(x,\underline{a}) = \frac{\sqrt{x}}{P_m(x,\underline{a})^{\frac{1}{4}}} e^{-z}\, \mathrm{s}_\alpha \psi_+(z,\underline{a})\,|_{z\,=\,z(x,\underline{a})}, \tag{40}$$

for x in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{m}-\frac{2\alpha}{m}, \frac{\pi}{m}-\frac{2\alpha}{m}[$, uniformly in $\underline{a}$ for $\underline{a}$ in any compact set of $\mathbb{C}^m$, where the direction of Borel resummation α runs through $]-\pi,+\pi[$.

The arguments used to prove proposition 3.16 can be extended to analyze the whole analytic structure of the minor $\widetilde{\psi}_+(\zeta,\underline{a})$ of $\psi_+(z,\underline{a})$. Since the techniques involved are the same as those used in [25] and [23], we just give the final result[5]:

Proposition 3.18. *The minor $\widetilde{\psi}_+(\zeta,\underline{a}) \in \mathbb{C}[\underline{a}]\{\zeta^{\frac{1}{m}}\}$ (resp. $\widetilde{\psi}_-(\zeta,\underline{a}) \in \mathbb{C}[\underline{a}]\{\zeta^{\frac{1}{m}}\}$) of $\psi_+(z,\underline{a})$ (resp. $\psi_-(z,\underline{a})$) can be extended analytically to $(\zeta,\underline{a}) \in \widetilde{\mathbb{C}\backslash\{0,-2\}} \times \mathbb{C}^m$ (resp. $(\zeta,\underline{a}) \in \widetilde{\mathbb{C}\backslash\{0,+2\}} \times \mathbb{C}^m$), where $\widetilde{\mathbb{C}\backslash\{0,\pm2\}}$ is the universal covering of $\mathbb{C}\backslash\{0,\pm2\}$. Moreover, $\widetilde{\psi}_\pm$ has an exponential growth of order not greater than 1 at infinity in ζ, uniformly in $\underline{a}$ for $\underline{a}$ in any given compact set of $\mathbb{C}^m$.*

One can make things more precise concerning the resurgent structure, that is the behavior of $\widetilde{\psi}_+$ and $\widetilde{\psi}_-$ near their singular points. To do that, we shall use the alien derivations.
We would like to compute $\Delta_\tau\psi_+$, where Δ_τ stands for the alien derivation at τ. From proposition 3.18 we know that the singular points of the minor of ψ_+ lie above -2 and 0. However, since ψ_+ belongs to $\mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]$, the non vanishing $\Delta_\tau\psi_+$ can be indexed by the elements τ above -2 on the Riemann surface $\mathfrak{C}_m$ of $z^{\frac{1}{m}}$.
We now use one of the fundamental properties of the alien derivations: the pointed alien derivation $\dot{\Delta}_\tau = e^{-\tau z}\Delta_\tau$ commutes with $\frac{d}{dz}$ (proposition 3.15). Using the fact that the resurgent symbol (definition 3.9) $\Psi_+(z,\underline{a}) = e^{-z}\psi_+(z,\underline{a})$ is solution of equation (14) and that F is a constant of resurgence (definition 3.14), we obtain:

$$-\frac{d^2}{dz^2}\left(\dot{\Delta}_\tau\Psi_+\right) + (1-F(z,\underline{a}))\left(\dot{\Delta}_\tau\Psi_+\right) = 0.$$

This means that $\dot{\Delta}_\tau\Psi_+$ satisfies the same equation (14). Since (Ψ_+,Ψ_-) is a fundamental system of formal solutions for (14), we can conclude that $\dot{\Delta}_\tau\Psi_+$ has to be proportional to the resurgent symbol Ψ_- by an argument of singular support (definition 3.9): the singular support of Ψ_+ (resp. Ψ_-) reduces to $\{+1\}$ (resp. $\{-1\}$), whereas, by definition, the resurgent symbol $\dot{\Delta}_\tau\Psi_+ = e^{+z} \times$ (a formal resurgent function) has $\{-1\}$ for its singular support. We deduce that there is $\delta_\tau(\underline{a})$ such that $\dot{\Delta}_\tau\Psi_+(z,\underline{a}) = \delta_\tau(\underline{a})\Psi_-(z,\underline{a})$, i.e., $\Delta_\tau\psi_+(z,\underline{a}) = \delta_\tau(\underline{a})\psi_-(z,\underline{a})$.

[5]For the particular reader, this part is detailed in Rasoamanana [32]

Similarly, one obtains the existence of $\delta_\tau(\underline{a})$ such that $\Delta_\tau\psi_-(z,\underline{a}) = \delta_\tau(\underline{a})\psi_+(z,\underline{a})$, where τ is above $+2$ on the Riemann surface $\mathfrak{C}_m$.

The coefficients $\delta_\tau(\underline{a})$ are entire functions of $\underline{a}$: this stems directly from the regularity in $\underline{a}$ of the formal series ψ_+, ψ_-, and from the fact that the location of the singular points of the minors does not depend on $\underline{a}$, so that the Stokes automorphism (in any direction) commutes with the analytic continuation in $\underline{a}$. However, this will be a consequence of theorem 3.27 which will be discussed in a moment.

To sum up:

Theorem 3.19. *For m odd, there exists a unique formal power series $\psi_+(z,\underline{a}) \in \mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]$ (resp. $\psi_-(z,\underline{a}) \in \mathbb{C}[\underline{a}][[z^{-\frac{1}{m}}]]$) whose residual coefficient is 1, such that $e^{-z}\psi_+(z,\underline{a})$ (resp. $e^{+z}\psi_-(z,\underline{a})$) is solution of equation (14), and moreover:*

$$\psi_-(z,\underline{a}) = \psi_+(\omega^{\frac{m}{2}}z, \omega.\underline{a}). \tag{41}$$

These formal power series $\psi_\pm$ are resurgent in z with holomorphic dependence in $\underline{a}$, and Borel-resummable[6] in z, uniformly with respect to $\underline{a}$ in any compact set.
Their resurgent structure is given by:

$$\begin{cases} \Delta_{2e^{ki\pi}}\psi_-(z,\underline{a}) = S_k(\underline{a})\psi_+(z,\underline{a}) & \text{for} \quad k \in 2\mathbb{Z} \\ \Delta_{2e^{ki\pi}}\psi_+(z,\underline{a}) = S_k(\underline{a})\psi_-(z,\underline{a}) & \text{for} \quad k-1 \in 2\mathbb{Z} \\ \Delta_\tau\psi_\pm = 0 & \text{otherwise,} \end{cases}$$

where Δ_τ is the alien derivation at τ. The coefficients $S_k(\underline{a})$ are entire functions in $\underline{a}$ and, for all $k \in \mathbb{Z}$, $S_k = S_{k \mod 2m}$.

Definition 3.20. The coefficients $S_k(\underline{a})$, $k \in \mathbb{Z}$, are called *the Stokes multipliers.*

3.2.2 The m even case

The fundamental difference with the previous m odd case is now the existence of the term $b_0(\underline{a})\ln(x)$ in the asymptotic expansion of $z(x,\underline{a})$ (defined by (12)) at infinity in x. This is why it is worth considering the following new Green-Liouville transformation

$$\begin{cases} \widetilde{z} = \widetilde{z}(x,\underline{a}) = \displaystyle\int^x \frac{\sqrt{P_m(t,\underline{a})} - b_0(\underline{a})}{t}dt \\ \\ \Psi(\widetilde{z},\underline{a}) = \dfrac{\sqrt{\sqrt{P_m(x,\underline{a})} - b_0(\underline{a})}}{\sqrt{x}}\Phi(x,\underline{a}) \end{cases} \tag{42}$$

so that the Laurent-Puiseux series expansion of $x \mapsto \widetilde{z}(x,\underline{a})$ coincides with the map $x \mapsto S(x,\underline{a})$ defined in theorem 2.1 modulo an analytic function vanishing at infinity. The quasi-homogeneity properties (13) and (16) are still valid for the maps $(x,\underline{a}) \mapsto \widetilde{z}(x,\underline{a})$ and $(\widetilde{z},\underline{a}) \mapsto x(\widetilde{z},\underline{a})$ respectively.

Equation $(\mathfrak{E}_m)$ is converted into the prepared equation :

$$-\frac{d^2}{d\widetilde{z}^2}\Psi + \left(1 + \frac{4b_0(\underline{a})}{m\widetilde{z}} - H(\widetilde{z},\underline{a})\right)\Psi = 0 \tag{43}$$

[6] Except of course for the singular directions which are described by the resurgence structure.

with

$$\left\{\begin{array}{l} H(\widetilde{z},a) = 1 + \dfrac{4b_0(\underline{a})}{m\widetilde{z}} - \dfrac{P_m(x,\underline{a})}{(\sqrt{P_m(x,\underline{a})} - b_0(\underline{a}))^2} \\ -\Big(\dfrac{xP_m'(x,\underline{a}) + P_m(x,\underline{a}) - b_0(\underline{a})\sqrt{P_m(x,\underline{a})}}{4\sqrt{P_m(x,\underline{a})}(\sqrt{P_m(x,\underline{a})} - b_0(\underline{a}))^3} \\ \qquad\qquad +x^2\Big(\dfrac{1}{4}\dfrac{P_m''(x,\underline{a})}{\sqrt{P_m(x,\underline{a})}(\sqrt{P_m(x,\underline{a})} - b_0(\underline{a}))^3} \\ \quad -\dfrac{1}{16}\dfrac{(P_m'(x,\underline{a}))^2(5\sqrt{P_m(x,\underline{a})} - 2b_0(\underline{a}))}{(P_m(x,\underline{a}) - b_0(\underline{a})\sqrt{P_m(x,\underline{a})})^3(\sqrt{P_m(x,\underline{a})} - b_0(\underline{a}))}\Big)\Big) |_{\widetilde{z} = \widetilde{z}(x,\underline{a})} \end{array}\right. \tag{44}$$

and

$$H(\widetilde{z},\underline{a}) = \frac{m^2-4}{4m^2\widetilde{z}^2}(1 + O(\widetilde{z}^{-\frac{2}{m}})) \in \frac{1}{\widetilde{z}^2}\mathbb{C}[\underline{a}]\{\widetilde{z}^{-\frac{2}{m}}\}. \tag{45}$$

Furthermore, H satisfies the quasi-homogeneity property (22).

One easily proves the existence of a unique formal solution

$$\Psi_+(\widetilde{z},\underline{a}) = e^{-\widetilde{z}}\psi_+(\widetilde{z},\underline{a})$$

of (43) satisfying

$$\psi_+(\widetilde{z},\underline{a}) = \widetilde{z}^{-\frac{2b_0(\underline{a})}{m}}\mu_+(\widetilde{z},\underline{a})$$

where $\mu_+ \in \mathbb{C}[\underline{a}][[\widetilde{z}^{-\frac{2}{m}}]]$ with residual coefficient 1. By quasi-homogeneity, one deduces the existence of another formal solution

$$\Psi_-(\widetilde{z},\underline{a}) = e^{+\widetilde{z}}\psi_-(\widetilde{z},\underline{a}) = e^{+\widetilde{z}}\widetilde{z}^{+\frac{2b_0(\underline{a})}{m}}\mu_-(\widetilde{z},\underline{a})$$

of (43) such that

$$\psi_-(\widetilde{z},\underline{a}) = \psi_+(\omega^{\frac{m}{2}}\widetilde{z},\omega.\underline{a}).$$

From now on the analysis is exactly the same as in the m odd case and yields the following results:

Theorem 3.21. *For m even, there exists a unique formal series $\psi_+(\widetilde{z},\underline{a}) = \widetilde{z}^{-\frac{2b_0(\underline{a})}{m}}\mu_+(\widetilde{z},\underline{a})$ where $\mu_+ \in \mathbb{C}[\underline{a}][[\widetilde{z}^{-\frac{2}{m}}]]$ with residual coefficient 1 (resp. $\psi_-(\widetilde{z},\underline{a}) = \widetilde{z}^{+\frac{2b_0(\underline{a})}{m}}\mu_-(\widetilde{z},\underline{a})$, $\mu_- \in \mathbb{C}[\underline{a}][[\widetilde{z}^{-\frac{2}{m}}]]$ with residual coefficient 1), such that $e^{-\widetilde{z}}\psi_+(\widetilde{z},\underline{a})$ (resp. $e^{+\widetilde{z}}\psi_-(\widetilde{z},\underline{a})$) is solution of equation (43). Moreover,*

$$\psi_-(\widetilde{z},\underline{a}) = \psi_+(\omega^{\frac{m}{2}}\widetilde{z},\omega.\underline{a}). \tag{46}$$

The formal power series $\psi_\pm$ are resurgent in $\widetilde{z}$ with holomorphic dependence in $\underline{a}$, and Borel-resummable in $\widetilde{z}$, uniformly with respect to $\underline{a}$ in any compact set.
There exists a set of entire functions $S_k(\underline{a})$, the Stokes multipliers, *such that :*

$$\left\{\begin{array}{ll} \Delta_{2e^{ki\pi}}\psi_-(\widetilde{z},\underline{a}) = S_k(\underline{a})\psi_+(\widetilde{z},\underline{a}) & \text{for} \quad k \in 2\mathbb{Z} \\ \Delta_{2e^{ki\pi}}\psi_+(\widetilde{z},\underline{a}) = S_k(\underline{a})\psi_-(\widetilde{z},\underline{a}) & \text{for} \quad k-1 \in 2\mathbb{Z} \\ \Delta_\tau\psi_\pm = 0 \qquad \text{otherwise,} & \end{array}\right.$$

where Δ_τ is the alien derivation at τ.

In this theorem, due to the fact that the formal solutions μ_+ and μ_- belong to $\mathbb{C}[\underline{a}][[z^{-\frac{2}{m}}]]$, the alien derivations need only to be indexed by elements on the Riemann surface of $z^{\frac{2}{m}}$. Thus *a priori* only m Stokes multipliers govern the resurgence structure. Nevertheless, it is better to describe the resurgence structure in terms of ψ_+ and ψ_-, which have to be thought of as formal functions on the universal covering of $\mathbb{C}^\star$.

Returning to theorem 2.1, we finally get the desired result:

Proposition 3.22. *When m is even, the analytic function Φ_0 of theorem 2.1 is given by*

$$\Phi_0(x,\underline{a}) = \frac{\sqrt{x}}{(\sqrt{P_m(x,\underline{a})} - b_0(\underline{a}))^{\frac{1}{2}}} e^{-\widetilde{z}}\, \mathrm{s}_\alpha \psi_+(\widetilde{z},\underline{a})\,|_{\widetilde{z}\,=\,\widetilde{z}(x,\underline{a})}, \tag{47}$$

for x in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{m}-\frac{2\alpha}{m}, \frac{\pi}{m}-\frac{2\alpha}{m}[$, uniformly in $\underline{a}$ for $\underline{a}$ in any compact set of $\mathbb{C}^m$, where the direction of Borel resummation α runs through $]-\pi,+\pi[$.

3.3 Some properties of the Stokes multipliers

The quasi homogeneity induces some interesting properties of the Stokes multipliers.

To simplify, we assume from now on that:

Notation 3.23.

$$\omega = e^{\frac{2i\pi}{m}}. \tag{48}$$

We recall the following easy result in resurgence theory, cf. [18].

Lemma 3.24. *Let $\psi_1(y)$ be a formal resurgent function and let ν be a nonzero complex number. Setting $y = \nu t$ and $\psi_2(t) = \psi_1(y)$, we have the following equality :*

$$\Delta^t_{\nu\tau}\psi_2 = \Delta^y_\tau \psi_1.$$

where Δ^x_τ denotes the alien derivation at τ with respect to the variable x.

Proposition 3.25. *With the notations of theorem 3.19 and 3.21, we have, for all $k \in \mathbb{Z}$:*

$$S_k(\underline{a}) = S_0(\omega^k.\underline{a}).$$

Proof. In theorems 3.19 and 3.21 we introduce $t = z$ for m odd, $t = \widetilde{z}$ for m even. From (41) and (46), we get

$$\begin{cases} \psi_+(e^{+i\pi}t, \omega.\underline{a}) = \psi_-(t,\underline{a}) \\ \\ \psi_+(t,\underline{a}) = \psi_-(e^{-i\pi}t, \omega^{-1}.\underline{a}). \end{cases}$$

Using lemma 3.24 with $y = e^{i\pi}t$, we deduce

$$\Delta^y_{2e^{i0}}\psi_-(y,\omega.\underline{a}) = \Delta^t_{2e^{i\pi}}\psi_+(t,\underline{a}).$$

Now, by definition of S_0 and S_1 (theorems 3.19 and 3.21), we have the equalities :

$$\begin{cases} \Delta^y_{2e^{i0}}\psi_-(y,\omega.\underline{a}) = S_0(\omega.\underline{a})\psi_+(y,\omega.\underline{a}) \\ \\ \Delta^z_{2e^{i\pi}}\psi_+(t,\underline{a}) = S_1(\underline{a})\psi_-(t,\underline{a}). \end{cases}$$

Finally, we obtain :

$$S_1(\underline{a}) = S_0(\omega.\underline{a}).$$

We end the proof by an easy induction argument. $\square$

Since $\omega^m = e^{2i\pi}$, we get:

Corollary 3.26. *For all $k \in \mathbb{Z}$,*

$$S_k = S_{k \mod m}.$$

3.4 Stokes-Sibuya coefficients and Stokes multipliers

To describe the connection formulas, we have now two sets of Stokes coefficients at our disposal. One is made up of the Stokes-Sibuya coefficients $C_k(\underline{a})$, the other is made up of the Stokes multipliers $S_k(\underline{a})$. The following proposition clarifies the relations between these two fundamental data.

Theorem 3.27. *We consider the Stokes-Sibuya coefficients C_k given by theorem 2.7 and the Stokes multipliers described by theorems 3.19 and 3.21. Then, for all $k \in \mathbb{Z}$,*

$$S_k(\underline{a}) = \omega^{r(\omega^k.\underline{a})} C_k(\underline{a}) \tag{49}$$

where ω is given by (48).

In this theorem, $r(\underline{a})$ has been defined in theorem 2.1. In particular, when m is odd, then $r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4}$ does not depend on $\underline{a}$, so that (49) simply reads

$$\text{for } m \text{ odd }, S_k(\underline{a}) = \omega^r C_k(\underline{a}).$$

Proof. To simplify, we give the proof for m odd only, so that $r = \dfrac{1}{2} - \dfrac{m}{4}$.

By proposition 3.25 and formula (7) of theorem 2.7, it is sufficient to show (49) for $k = 0$. By proposition 3.17, Φ_0 of theorem 2.1 can be defined by

$$\Phi_0(x, \underline{a}) = \frac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} e^{-z}\, \mathrm{s}_0\psi_+(z, \underline{a}) \,|_{z\, =\, z(x, \underline{a})}, \tag{50}$$

for z in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{2}, \frac{\pi}{2}[$, which corresponds to x in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{m}, \frac{\pi}{m}[$.
Now, by theorem 2.7, Φ_1 is defined by

$$\Phi_1(x, \underline{a}) = \Phi_0(\omega x, \omega.\underline{a}).$$

Using (50), we get the following representation for Φ_1 :

$$\Phi_1(x, \underline{a}) = \frac{\sqrt{x\omega}}{P_m(x\omega, \omega.\underline{a})^{\frac{1}{4}}} e^{-z} \mathrm{s}_0\psi_+(z, \omega.\underline{a}) \,|_{z\, =\, z(\omega x, \omega.\underline{a})}, \tag{51}$$

for ωx in a sectorial neighbourhood of infinity of aperture $]-\frac{\pi}{m}, \frac{\pi}{m}[$, i.e., x in a sectorial neighbourhood of infinity of aperture $]-\frac{3\pi}{m}, -\frac{\pi}{m}[$, so that z belongs to a sectorial neighbourhood of infinity of aperture $]-\frac{3\pi}{2}, -\frac{\pi}{2}[$. By quasi-homogeneity considerations, we have seen that $z(\omega x, \omega.\underline{a}) = e^{i\pi} z(x, \underline{a})$, (cf. formula (13)), so that, by formula (41) of theorem 3.19,

$$\psi_+(z(\omega x, \omega.\underline{a}), \omega.\underline{a}) = \psi_-(z, \underline{a}).$$

Also,

$$P_m(\omega x, \omega.\underline{a}) = \omega^m P_m(x, \underline{a}).$$

This means that (51) can be written as

$$\Phi_1(x, \underline{a}) = \omega^r \frac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} e^z \mathrm{s}_\pi \psi_-(z, \underline{a})\,|_{z\,=\,z(x,\underline{a})}$$

for x (*resp.* z) in a sectorial neighbourhood of infinity of aperture $]-\frac{3\pi}{m}, -\frac{\pi}{m}[$ (*resp.* $]-\frac{3\pi}{2}, -\frac{\pi}{2}[$). As for Φ_1, we have the following representation for Φ_{-1} :

$$\Phi_{-1}(x, \underline{a}) = \omega^{-r} \frac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} e^z \mathrm{s}_{-\pi} \psi_-(z, \underline{a})\,|_{z\,=\,z(x,\underline{a})}$$

for x (*resp.* z) in a sectorial neighbourhood of infinity of aperture $]\frac{\pi}{m}, \frac{3\pi}{m}[$ (*resp.* $]\frac{\pi}{2}, \frac{3\pi}{2}[$).

To compare Φ_{-1}, Φ_0, and Φ_1, we rotate the directions of resummation so as to sum along the direction 0. Since $\Delta_{2e^{i0}}\psi_-(z, \underline{a}) = S_0(\underline{a})\psi_+(z, \underline{a})$ (cf. theorem 3.19), ψ_- is not Borel-resummable in the direction 0 if $S_0(\underline{a}) \neq 0$, but only right or left Borel-resummable. In other words, we have to take into account a Stokes phenomenon. Since $\underline{\dot{\Delta}}_0\psi_-(z, \underline{a})$ (definition 3.11) reduces to $\dot{\Delta}_{2e^{i0}}\psi_-(z, \underline{a})$, one gets $\mathfrak{S}_0\psi_-(z, \underline{a}) = \psi_-(z, \underline{a}) + \dot{\Delta}_{2e^{i0}}\psi_-(z, \underline{a})$, where $\mathfrak{S}_0$ is the Stokes automorphism in the direction 0 (definition 3.10). Therefore, $\mathrm{s}_{0-}\psi_-(z, \underline{a}) = \mathrm{s}_{0+}\left[\psi_-(z, \underline{a}) + e^{-2z}S_0(\underline{a})\psi_+(z, \underline{a})\right]$, where s_{0+} (resp. s_{0-}) is the right (resp. left) Borel-resummation in the direction 0.

We obtain:

$$\begin{cases} \Phi_0(x, \underline{a}) = \dfrac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} e^{-z} \mathrm{s}_0 \psi_+(z, \underline{a})\,|_{z\,=\,z(x,\underline{a})} \\[2ex] \Phi_1(x, \underline{a}) = \omega^r \dfrac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} e^{+z} \mathrm{s}_{0+} \psi_-(z, \underline{a})\,|_{z\,=\,z(x,\underline{a})} \\[2ex] \Phi_{-1}(x, \underline{a}) = \omega^{-r} \dfrac{\sqrt{x}}{P_m(x, \underline{a})^{\frac{1}{4}}} \mathrm{s}_{0+}[e^z\psi_-(z, \underline{a}) + e^{-z}S_0(\underline{a})\psi_+(z, \underline{a})]\,|_{z\,=\,z(x,a)}. \end{cases} \tag{52}$$

By theorem 2.7, we have the connection formula $\Phi_{-1} = C_0(\underline{a})\Phi_0 + \widetilde{C}_0(\underline{a})\Phi_1$; in this equality, replacing Φ_{-1}, Φ_0, Φ_1 by the right-hand sides of (52) and equating the coefficients of $e^{-z}\mathrm{s}_0\psi_+$ and $e^{+z}\mathrm{s}_{0+}\psi_-$, we finally get:

$$\begin{cases} S_0(\underline{a}) = \omega^r C_0(\underline{a}) \\[1ex] \widetilde{C}_0(\underline{a}) = \omega^{-2r} = \omega^{m-2+2r}. \end{cases}$$

□

4 Solutions of $(\mathfrak{E}_m)$ in the neighbourhood of the origin: Fuchs theory

In order to get more information about the Stokes-Sibuya coefficients C_k (or about the Stokes multipliers S_k, since this is equivalent, by theorem 3.27), we have to pick up the necessary information coming from the other singular point of $(\mathfrak{E}_m)$, i.e., the origin.

Since the origin is a regular singular point of $(\mathfrak{E}_m)$, the classical Fuchs theory allows to describe "canonical" systems of solutions of $(\mathfrak{E}_m)$ near the origin (see, e.g., [33, 42]). The characteristic equation is $s(s-1)-a_m=0$, so that $\dfrac{1\pm p}{2}$ are the characteristic values, with $p=(1+4a_m)^{\frac{1}{2}}$.

Notation 4.1. In what follows, $p=(1+4a_m)^{\frac{1}{2}}$ and $s(p)=\dfrac{1+p}{2}$.
We note $\underline{a}' := (a_1,\cdots,a_{m-1})$, and for all $\tau\in\mathbb{C}$,

$$\tau.\underline{a}' := (\tau a_1,\cdots,\tau^{m-1}a_{m-1}).$$

As is well-known, we have to distinguish between the following three cases : $p\notin\mathbb{Z}$, $p\in\mathbb{Z}^\star$ and $p=0$. Since we have the freedom for choosing the determination of the square root $(1+4a_m)^{\frac{1}{2}}$, we can avoid the case where $p\in-\mathbb{N}^\star$ in the following theorem.

Theorem 4.2. *There exist two unique linearly independent solutions f_1, f_2 of $(\mathfrak{E}_m)$ such that*

$$\begin{cases} f_1(x,\underline{a}',p)=x^{s(p)}g_1(x,\underline{a}',p)=x^{s(p)}\left(1+\displaystyle\sum_{k=1}^{\infty}A_k(\underline{a}',p)x^k\right) \\ \\ f_2(x,\underline{a}',p)=\lambda(\underline{a}',p)f_1(x,\underline{a}',p)\ln(x)+x^{s(-p)}g_2(x,\underline{a}',p)\end{cases}$$

where g_1, g_2 are entire functions in x and $\underline{a}'$, while λ is entire in $\underline{a}'$. Moreover, g_1 is meromorphic in p with at most simple poles when $-p\in\mathbb{N}^\star$. Precisely, for all $k\in\mathbb{N}^\star$, $A_k(\underline{a}',p)\displaystyle\prod_{l=1}^{k}(p+l)\in\mathbb{C}[\underline{a}',p]$.

1. *When $p\notin\mathbb{Z}$, then $\lambda(\underline{a}',p)=0$ and $g_2(x,\underline{a}',p)=g_1(x,\underline{a}',-p)$.*
2. *When $p\in\mathbb{N}^\star$, then*

$$g_2(x,\underline{a}',p)=\left(1+\sum_{k=1}^{\infty}B_k(\underline{a}',p)x^k\right)\qquad \textit{with}\qquad B_p=0.$$

   Moreover, for all $k\in\mathbb{N}^\star$, $\lambda(\underline{a}',p)$, $B_k(\underline{a}',p)\in\mathbb{C}[\underline{a}']$, and

$$\lambda(\omega.\underline{a}',p)=\omega^{-p}\lambda(\underline{a}',p).$$

3. *When $p=0$, then $\lambda(\underline{a}',p)=1$ and*

$$g_2(x,\underline{a}',p)=\sum_{k=1}^{\infty}B_k(\underline{a}',p)x^k$$

   with, for all $k\in\mathbb{N}^\star$, $B_k(\underline{a}',p)\in\mathbb{C}[\underline{a}']$.

Remark 4.3. When $-p\in\mathbb{N}^\star$, just change p into $-p$ in theorem 4.2, which corresponds to choosing the other root for $(1+4a_m)^{\frac{1}{2}}$, or equivalently, which corresponds to a_m making a loop around $-1/4$.

Remark 4.4. In the special case when $\underline{a}' = 0$, the function g_1 is meromorphic in p with at most simple poles when $-p \in m\mathbb{N}^\star$.

The existence and unicity of f_1 and f_2 follow from the Fuchs theory, and the chosen normalization for g_1 and g_2. The properties of the coefficients A_k, B_k and λ can be proved by induction, and this induces the analytic properties of g_1, g_2. The quasi-homogeneity property of λ is a consequence of the quasi-homogeneity of equation $(\mathfrak{E}_m)$.

The following result can be shown also by induction, see Rasoamanana [32]:

Proposition 4.5. *We consider $p \in \mathbb{N}^\star$ and we note $p = km + r$, $0 \le r \le m-1$ its Euclidian division by m. We introduce $\epsilon(r) = \begin{cases} 1 \text{ if } r \neq 0 \\ 0 \text{ if } r = 0 \end{cases}$. Then*

$$\lambda(\underline{a}', p) =$$

$$\frac{1}{p} \sum_{l=k+\epsilon(r)}^{p} \sum_{\substack{i_1+\cdots+i_l=p \\ 1\le i_j \le m}} \frac{a_{m-i_1} \cdots a_{m-i_l}}{i_1(i_1-p)\cdots(i_1+\cdots+i_{l-1})(i_1+\cdots+i_{l-1}-p)}$$

with the convention $a_0 = 1$.

Remark 4.6. For $p \in \mathbb{N}^\star$,

- in the special case when $\underline{a}' = 0$, then

$$\lambda(0,p)\,|_{p\neq 0 \mod m} = 0$$

  while, for $k \in \mathbb{N}^\star$,

$$\lambda(0,p)\,|_{p=km} = \frac{(-1)^{k+1}}{m^{2k-1} k \Gamma(k)^2}.$$

- When $m = 2$, then

$$\begin{cases} \lambda(\underline{a}', p)\,|_{p=0 \mod 2} = -\dfrac{1}{p\Gamma(p)^2} \displaystyle\prod_{k=1}^{p/2} (a_1 + 2k - 1)(a_1 - 2k + 1) \\[2ex] \lambda(\underline{a}', p)\,|_{p=1 \mod 2} = \dfrac{1}{p\Gamma(p)^2} a_1 \displaystyle\prod_{k=1}^{(p-1)/2} (a_1 + 2k)(a_1 - 2k) \end{cases}.$$

From the uniqueness of f_1 and f_2 in theorem 4.2, and from the quasi-homogeneity of equation $(\mathfrak{E}_m)$, we easily obtain:

Corollary 4.7. *We consider the fundamental system of solutions (f_1, f_2) of theorem 4.2. Then,*

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (\omega x, \omega.\underline{a}', p) = \mathfrak{N}(\underline{a}', p) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (x, \underline{a}', p) \tag{53}$$

with

$$\mathfrak{N}(\underline{a}', p) = \begin{pmatrix} \omega^{s(p)} & 0 \\ \frac{2i\pi}{m}\lambda(\underline{a}', p)\omega^{s(-p)} & \omega^{s(-p)} \end{pmatrix}. \tag{54}$$

Moreover

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\omega^m x, \underline{a}', p) = \mathfrak{M}(\underline{a}', p) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x, a) \tag{55}$$

where

$$\mathfrak{M}(\underline{a}', p) = \begin{pmatrix} e^{2i\pi s(p)} & 0 \\ 2i\pi\lambda(\underline{a}', p)e^{2i\pi s(-p)} & e^{2i\pi s(-p)} \end{pmatrix} \tag{56}$$

is the monodromy matrix at the origin.

5 The 0∞ connection matrices

In section 2, we described a set of fundamental systems of solutions (Φ_{k-1}, Φ_k) of $(\mathfrak{E}_m)$, where $k \in \mathbb{Z}$. In section 4, we have obtained another fundamental system of solutions (f_1, f_2). To compare these fundamental systems, we introduce, for all $k \in \mathbb{Z}$:

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a}) = M_k(\underline{a}', p) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x, \underline{a}', p) \tag{57}$$

where the matrices $M_k(\underline{a}', p)$ are invertible.

Definition 5.1. The matrices $M_k(\underline{a}', p)$ are called the 0∞-*connection matrices*.

We now give some properties of the M_k. These properties depend essentially on p, as does the fundamental system (f_1, f_2).

We start with an obvious result.

Proposition 5.2. *For every* $k \in \mathbb{Z}$,

$$M_{k+1}(\underline{a}', p) = M_k(\omega.\underline{a}', p)\mathfrak{N}(\underline{a}', p) \tag{58}$$

where the invertible matrix $\mathfrak{N}(\underline{a}', p)$ *is given by (54).*

Proof. By theorem 2.7, we write

$$\begin{pmatrix} \Phi_k \\ \Phi_{k+1} \end{pmatrix}(x, \underline{a}) = \begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(\omega x, \omega.\underline{a}) = M_k(\omega.\underline{a}', p) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\omega x, \omega.\underline{a}', p).$$

Since, by definition of $\mathfrak{N}$, $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\omega x, \omega.\underline{a}', p) = \mathfrak{N}(\underline{a}', p) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x, \underline{a}', p)$, we can conclude because (f_1, f_2) is a fundamental system. □

Theorem 5.3. *a) For every* $k \in \mathbb{Z}$,

$$\det M_k(\underline{a}', p) = \begin{cases} 2(-1)^k \dfrac{\omega^{(k-1)(1-\frac{m}{2})+r(\omega^k.\underline{a})}}{p} & \text{for } p \neq 0 \\ 2(-1)^{k-1}\omega^{(k-1)(1-\frac{m}{2})+r(\omega^k.\underline{a})} & \text{for } p = 0. \end{cases} \tag{59}$$

b) For every $k \in \mathbb{Z}$, *the matrix* $M_k(\underline{a}', p)$ *is entire in* $\underline{a}'$. *More precisely,*

$$M_k(\underline{a}', p) = \begin{pmatrix} L_k(\underline{a}', p) & \widetilde{L}_k(\underline{a}', p) \\ \omega^{s(p)}L_k(\omega.\underline{a}', p) + \frac{2i\pi}{m}\lambda(\underline{a}', p)\omega^{s(-p)}\widetilde{L}_k(\omega.\underline{a}', p) & \omega^{s(-p)}\widetilde{L}_k(\omega.\underline{a}', p) \end{pmatrix} \tag{60}$$

where $L_k(\underline{a}', p)$ *and* $\widetilde{L}_k(\underline{a}', p)$ *are entire in* $\underline{a}'$.

c) For every $k \in \mathbb{Z}$, *the matrix* $M_k(\underline{a}', p)$ *is holomorphic in* $p \notin \mathbb{Z}$, *and*

$$\forall p \notin \mathbb{Z}, \forall \underline{a}' \in \mathbb{C}^{m-1}, \ \widetilde{L}_k(\underline{a}', p) = L_k(\underline{a}', -p).$$

Moreover, $\widetilde{L}_k$ *extends analytically at* $p \in \mathbb{N}^\star$.

d) For every $k \in \mathbb{Z}$,

$$M_k(\underline{a}', p) = M_0(\omega^k.\underline{a}', p) \begin{pmatrix} \omega^{ks(p)} & 0 \\ \frac{2i\pi k}{m}\lambda(\underline{a}', p)\omega^{ks(-p)} & \omega^{ks(-p)} \end{pmatrix}. \tag{61}$$

In particular,

$$M_m(\underline{a}', p) = M_0(\underline{a}', p)\mathfrak{M}(\underline{a}', p). \tag{62}$$

Proof. We only detail the proof for $p \notin \mathbb{Z}$.

a) We deduce from (57) that

$$W(\Phi_{k-1}, \Phi_k) = \det(M_k) W(f_1, f_2)$$

where $W(.,.)$ is the Wronskian. From lemma 2.5, we know that

$$W(\Phi_{k-1}, \Phi_k) = 2(-1)^{k-1}\omega^{(k-1)(1-\frac{m}{2})+r(\omega^k.\underline{a})},$$

while, by theorem 4.2, taking the limit $x \to 0$ and using the fact that the wronskian is x-independent, one easily gets

$$W(f_1, f_2) = s(-p) - s(p) = -p.$$

b) From (57), we have $\forall k \in \mathbb{Z}$, $\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(x, \underline{a}) = M_k(\underline{a}', p)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x, \underline{a}', p)$ with $M_k(\underline{a}', p) = \begin{pmatrix} \beta_{k1}(\underline{a}', p) & \beta_{k2}(\underline{a}', p) \\ \beta_{k3}(\underline{a}', p) & \beta_{k4}(\underline{a}', p) \end{pmatrix}$ so that, in particular,

$$\Phi_k(x, \underline{a}) = \beta_{k3}(\underline{a}', p) f_1(x, \underline{a}', p) + \beta_{k4}(\underline{a}', p) f_2(x, \underline{a}', p). \tag{63}$$

Then

$$\begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix}(\omega x, \omega.\underline{a}) = M_k(\omega.\underline{a}', p)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\omega x, \omega.\underline{a}', p).$$

By proposition 5.2 and corollary 4.7, we get:

$$\begin{pmatrix} \Phi_k \\ \Phi_{k+1} \end{pmatrix}(x, \underline{a}) = M_k(\omega.\underline{a}', p)\begin{pmatrix} \omega^{s(p)} & 0 \\ 0 & \omega^{s(-p)} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(x, \underline{a}', p)$$

so that

$$\Phi_k(x, \underline{a}) = \omega^{s(p)}\beta_{k1}(\omega.\underline{a}', p) f_1(x, \underline{a}', p) + \omega^{s(-p)}\beta_{k2}(\omega.\underline{a}', p) f_2(x, \underline{a}', p). \tag{64}$$

Comparing (63) and (64), we obtain the announced form for M_k with $\beta_{k1} = L_k$ and $\beta_{k2} = \widetilde{L}_k$, since (f_1, f_2) is a fundamental system.

b) and c) We have

$$M_k = -\frac{1}{p}\begin{pmatrix} \Phi_{k-1} & \Phi'_{k-1} \\ \Phi_k & \Phi'_k \end{pmatrix}\begin{pmatrix} f'_2 & -f'_1 \\ -f_2 & f_1 \end{pmatrix}$$

so that the analytic properties of M_k easily follow from the analytic properties of the Φ_k's (theorem 2.7) and of f_1, f_2 (theorem 4.2).

d) The given statement follows from proposition 5.2, by induction, inferring from (54) that $\mathfrak{N}(\omega.\underline{a}', p) = \mathfrak{N}(\underline{a}', p)$. □

In addition to theorem 5.3, it is easy to show the following proposition (the special case where $\underline{a}' = 0$ follows from remark 4.4):

Proposition 5.4. *The restriction to $p \notin \mathbb{Z}$ of the function $L_k(\underline{a}', p)$ (resp. $\widetilde{L}_k(\underline{a}', p)$) has a meromorphic continuation in p, with at most simple poles when $p \in \mathbb{N}$ (resp. $-p \in \mathbb{N}$).*
In the special case where $\underline{a}' = 0$, the restriction to $p \notin \mathbb{Z}$ of the function $L_k(\underline{a}', p)$ (resp. $\widetilde{L}_k(\underline{a}', p)$) has a meromorphic continuation in p, with at most simple poles at $p \in m\mathbb{N}$ (resp. $-p \in m\mathbb{N}$).

6 Monodromy, Stokes-Sibuya and 0∞ connection matrices

We collect here the different results we have got on the monodromy, the Stokes-Sibuya and the 0∞ connection matrices to obtain a set of functional relations.

6.1 First functional equation

From the very definition (57) of the 0∞ connection matrices M_k and the fundamental property (5) of the Stokes-Sibuya connection matrices, we have for all $k \in \mathbb{Z}$:

$$M_k \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \Phi_{k-1} \\ \Phi_k \end{pmatrix} = \mathfrak{S}_k \begin{pmatrix} \Phi_k \\ \Phi_{k+1} \end{pmatrix} = \mathfrak{S}_k M_{k+1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Since (f_1, f_2) is a fundamental system, we thus have the following proposition:

Proposition 6.1. *For all $k \in \mathbb{Z}$,*

$$\mathfrak{S}_k(\underline{a}) = M_k(\underline{a}', p) M_{k+1}^{-1}(\underline{a}', p). \tag{65}$$

Using (65), we see that

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) = M_0(\underline{a}', p) M_m^{-1}(\underline{a}', p).$$

Using (62), we obtain the following theorem:

Theorem 6.2. *The Stokes-Sibuya connection matrices satisfy the following functional relation:*

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a}) \cdots \mathfrak{S}_{m-1}(\underline{a}) = M_0(\underline{a}', p)\mathfrak{M}^{-1}(\underline{a}', p) M_0^{-1}(\underline{a}', p). \tag{66}$$

This functional relation is equivalent to formula (10) of theorem 2.9. But this new formulation is interesting thanks to the following two corollaries.

Corollary 6.3. *We have*

$$Tr\left(\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a})\right) = -2\cos(\pi p)$$

where Tr is the Trace.

Proof. This follows from the fact that $Tr\left(M_0(\underline{a}',p)\mathfrak{M}^{-1}(\underline{a}',p)M_0^{-1}(\underline{a}',p)\right) = Tr\left(\mathfrak{M}^{-1}(\underline{a}',p)\right) = -2\cos(\pi p)$. □

We have also the following result:

Corollary 6.4. *We assume that $p \in \mathbb{N}^\star$. Then, with the notations of theorem 4.2,*

$$\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a})\cdots\mathfrak{S}_{m-1}(\underline{a})\,|_{\lambda(\underline{a}',p)=0} = (-1)^{p+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. From corollary 4.7, we know that $\mathfrak{M}(\underline{a}',p) = e^{2i\pi s(p)}\begin{pmatrix} 1 & 0 \\ 2i\pi\lambda(\underline{a}',p) & 1 \end{pmatrix}$ with $2s(p) = 1+p$, so that

$$M_0(\underline{a}',p)\mathfrak{M}^{-1}(\underline{a}',p)M_0^{-1}(\underline{a}',p)\,|_{\lambda(\underline{a}',p)=0}$$

$$= \mathfrak{M}^{-1}(\underline{a}',p)\,|_{\lambda(\underline{a}',p)=0} = (-1)^{p+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

6.2 Second functional equation

Theorem 6.5. *We use the notations of theorem 5.3.*

- *1. We assume $p \notin \mathbb{Z}$. We assume furthermore that $\underline{a}$ is chosen so that, for all $k = 0,\cdots,m-1$, $\widetilde{L}_0(\omega^k.\underline{a}',p) \neq 0$[7]. Then*

$$\frac{L_0(\underline{a}',p)}{\widetilde{L}_0(\underline{a}',p)} = -i\frac{\omega^{-\frac{3}{2}}\omega^{-(m+1)\frac{p}{2}}}{p\sin(\pi p)}\sum_{k=0}^{m-1}\frac{\omega^{r(\omega^k.\underline{a})+(k+1)p}}{\widetilde{L}_0(\omega^k.\underline{a}',p)\widetilde{L}_0(\omega^{k+1}.\underline{a}',p)}. \tag{67}$$

- *2. We assume $p \in \mathbb{N}^\star$. Assuming also that $\underline{a}$ is chosen so that, for all $k = 0,\cdots,m-1$, $\widetilde{L}_0(\omega^k.\underline{a}',p) \neq 0$, then*

$$i\pi p\omega^{\frac{3}{2}+\frac{p}{2}}\lambda(\underline{a}',p) = \sum_{k=0}^{m-1}\frac{\omega^{r(\omega^k.\underline{a},p)+(k+1)p}}{\widetilde{L}_0(\omega^k.\underline{a}',p)\widetilde{L}_0(\omega^{k+1}.\underline{a}',p)}. \tag{68}$$

Proof. • 1. Using formulas (59) and (60) with $k = 0$, we get

$$\omega^{s(-p)}L_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p) - \omega^{s(p)}L_0(\omega.\underline{a}',p)\widetilde{L}_0(\underline{a}',p) = -\frac{2}{p}\omega^{-1+r(\underline{a})}$$

[7]Note that $\widetilde{L}_0(\underline{a}',p)$ cannot be identically zero; therefore, this is a generic hypothesis on $\underline{a}$.

and, more generally, for all $k = 0, \cdots, m-1$,

$$\omega^{s(-p)} L_0(\omega^k.\underline{a}', p)\widetilde{L}_0(\omega^{k+1}.\underline{a}', p) - \omega^{s(p)} L_0(\omega^{k+1}.\underline{a}', p)\widetilde{L}_0(\omega^k.\underline{a}', p) =$$

$$-\frac{2}{p}\omega^{-1+r(\omega^k.\underline{a})}.$$

We assume $\underline{a}$ generic so that for all $k = 0, \cdots, m-1$, $\widetilde{L}_0(\omega^k.\underline{a}', p) \neq 0$. The previous equalities read also:

$$\omega^{s(-p)}\frac{L_0(\omega^k.\underline{a}', p)}{\widetilde{L}_0(\omega^k.\underline{a}', p)} - \omega^{s(p)}\frac{L_0(\omega^{k+1}.\underline{a}', p)}{\widetilde{L}_0(\omega^{k+1}.\underline{a}', p)} = -\frac{2}{p}\frac{\omega^{-1+r(\omega^k.\underline{a})}}{\widetilde{L}_0(\omega^k.\underline{a}', p)\widetilde{L}_0(\omega^{k+1}.\underline{a}', p)}.$$

From the holomorphy in $\underline{a}'$ of L_0 and $\widetilde{L}_0$, this can be written in the following form, since $\omega^m = e^{2i\pi}$:

$$\mathfrak{L}\begin{pmatrix} \dfrac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} \\ \vdots \\ \dfrac{L_0(\omega^{m-1}.\underline{a}', p)}{\widetilde{L}_0(\omega^{m-1}.\underline{a}', p)} \end{pmatrix} = -\frac{2\omega^{-1}}{p}\begin{pmatrix} \dfrac{\omega^{r(\underline{a})}}{\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)} \\ \vdots \\ \dfrac{\omega^{r(\omega^{m-1}.\underline{a})}}{\widetilde{L}_0(\omega^{m-1}.\underline{a}', p)\widetilde{L}_0(\underline{a}', p)} \end{pmatrix},$$

where

$$\mathfrak{L} = \begin{pmatrix} \omega^{s(-p)} & -\omega^{s(p)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{s(-p)} & -\omega^{s(p)} \\ -\omega^{s(p)} & 0 & \cdots & 0 & \omega^{s(-p)} \end{pmatrix}$$

is a $m \times m$ circulant matrix whose determinant is $\omega^{ms(-p)} - \omega^{ms(p)}$. This determinant does not vanish because $s(p) - s(-p) = p$ is not an integer. The inverse of this matrix is also a circulant matrix, precisely:

$$\mathfrak{L}^{-1} = \frac{1}{\omega^{ms(-p)} - \omega^{ms(p)}} \times$$

$$\begin{pmatrix} \omega^{(m-1)s(-p)} & \omega^{(m-2)s(-p)+s(p)} & \cdots & \omega^{s(-p)+(m-2)s(p)} & \omega^{(m-1)s(p)} \\ \omega^{(m-1)s(p)} & \omega^{(m-1)s(-p)} & & \cdots & \omega^{s(-p)+(m-2)s(p)} \\ & \cdots & \cdots & \cdots & \\ & \cdots & \cdots & \cdots & \\ \omega^{(m-2)s(-p)+s(p)} & \cdots & & \omega^{(m-1)s(p)} & \omega^{(m-1)s(-p)} \end{pmatrix}.$$

This yields, since $s(p) - s(-p) = p$,

$$\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} = -\frac{2\omega^{-1}\omega^{(m-1)s(-p)}}{p(\omega^{ms(-p)} - \omega^{ms(p)})}\sum_{l=0}^{m-1}\omega^{lp}\frac{\omega^{r(\omega^l.\underline{a})}}{\widetilde{L}_0(\omega^l.\underline{a}', p)\widetilde{L}_0(\omega^{l+1}.\underline{a}', p)}$$

that is also

$$\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} = i\frac{\omega^{-1}\omega^{(m-1)s(-p)}}{p\sin(\pi p)}\sum_{l=0}^{m-1}\omega^{lp}\frac{\omega^{r(\omega^l.\underline{a})}}{\widetilde{L}_0(\omega^l.\underline{a}', p)\widetilde{L}_0(\omega^{l+1}.\underline{a}', p)}.$$

- 2. We work with formulas (59) and (60) with $k=0$, when $p\in\mathbb{N}^\star$. Using also the fact that $\lambda(\omega.\underline{a}',p)=\omega^{-p}\lambda(\underline{a}',p)$ (see theorem 4.2), we get

$$\mathfrak{L}\begin{pmatrix}\dfrac{L_0(\underline{a}',p)}{\widetilde{L}_0(\underline{a}',p)}\\ \vdots\\ \dfrac{L_0(\omega^{m-1}.\underline{a}',p)}{\widetilde{L}_0(\omega^{m-1}.\underline{a}',p)}\end{pmatrix}=$$

$$-\frac{2\omega^{-1}}{p}\begin{pmatrix}\dfrac{\omega^{r(\underline{a})}}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)}\\ \vdots\\ \dfrac{\omega^{r(\omega^{m-1}.\underline{a})}}{\widetilde{L}_0(\omega^{m-1}.\underline{a}',p)\widetilde{L}_0(\underline{a}',p)}\end{pmatrix}+\frac{2i\pi}{m}\omega^{s(-p)}\lambda(\underline{a}',p)\begin{pmatrix}1\\ \vdots\\ \omega^{-(m-1)p}\end{pmatrix}$$

where $\mathfrak{L}$ is the previous circulant matrix. But now $\det(\mathfrak{L})=\omega^{ms(-p)}-\omega^{ms(p)}=0$, since $s(p)-s(-p)=p\in\mathbb{N}^\star$. It is straightforward to see that $\mathfrak{L}$ is of rank $m-1$, so that the compatibilty condition reads

$$\det\begin{pmatrix}\omega^{s(-p)} & -\omega^{s(p)} & 0 & \cdots & 0 & \alpha_0\\ 0 & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & \omega^{s(-p)} & -\omega^{s(p)} & \\ 0 & 0 & \cdots & 0 & \omega^{s(-p)} & \alpha_{m-2}\\ -\omega^{s(p)} & 0 & \cdots & 0 & 0 & \alpha_{m-1}\end{pmatrix}=0$$

where

$$\alpha_k=-\frac{2\omega^{-1}}{p}\frac{\omega^{r(\omega^k.\underline{a})}}{\widetilde{L}_0(\omega^k.\underline{a}',p)\widetilde{L}_0(\omega^{k+1}.\underline{a}',p)}+\frac{2i\pi}{m}\omega^{s(-p)}\omega^{-kp}\lambda(\underline{a}',p).$$

This means that

$$\omega^{(m-1)s(-p)}\alpha_{m-1}+\omega^{(m-1)s(p)}\alpha_{m-2}+\omega^{(m-2)s(p)+s(-p)}\alpha_{m-3}+\\ \cdots+\omega^{s(p)+(m-2)s(-p)}\alpha_0=0,$$

that is also, because $s(p)-s(-p)=p$,

$$\alpha_{m-1}+\omega^{(m-1)p}\alpha_{m-2}+\omega^{(m-2)p}\alpha_{m-3}+\cdots+\omega^p\alpha_0=0.$$

We eventually get, since p is an integer:

$$i\pi p\omega^{\frac{3}{2}+\frac{p}{2}}\lambda(\underline{a}',p)=\frac{\omega^{r(\omega^{m-1}.\underline{a})}}{\widetilde{L}_0(\omega^{m-1}.\underline{a}',p)\widetilde{L}_0(\underline{a}',p)}+\sum_{k=0}^{m-2}\frac{\omega^{r(\omega^k.\underline{a}',p)+(k+1)p}}{\widetilde{L}_0(\omega^k.\underline{a}',p)\widetilde{L}_0(\omega^{k+1}\underline{a}',p)}.$$

□

Theorem 6.5 induces the following interesting result.

Corollary 6.6. *The Stokes-Sibuya multiplier $C_0(\underline{a})$ satisfies:*

- *when $m = 1$, for all $\underline{a} \in \mathbb{C}$:*
$$C_0(\underline{a}) = -2\cos(\pi p),$$

- *when $m = 2$, for all $\underline{a}' \in \mathbb{C}$ and $p \notin -\mathbb{N}$:*
$$C_0(\underline{a})\widetilde{L}_0(\omega.\underline{a}', p) = -2ie^{-i\pi\frac{a_1}{2}}\cos\left(\frac{\pi}{2}(p + a_1)\right)\widetilde{L}_0(\underline{a}', p),$$

- *when $m \geq 3$, for all $\underline{a}' \in \mathbb{C}^{m-1}$ and $p \notin -\mathbb{N}$:*
$$C_0(\underline{a})\widetilde{L}_0(\omega.\underline{a}', p) =$$
$$\omega^{r(\underline{a})-1+\frac{m}{4}}\left(\widetilde{L}_0(\underline{a}', p)\omega^{-r(\underline{a})+\frac{1}{2}-\frac{m}{4}+\frac{p}{2}} + \widetilde{L}_0(\omega^2.\underline{a}', p)\omega^{r(\underline{a})-\frac{1}{2}+\frac{m}{4}-\frac{p}{2}}\right).$$

Proof. Formula (65) of proposition 6.1 with $k = 0$ yields $\mathfrak{S}_0(\underline{a}) = M_0(\underline{a}', p)M_1^{-1}(\underline{a}', p)$. Using (60) and (61), we obtain:

$$\begin{pmatrix} C_0(\underline{a}) & \widetilde{C}_0(\underline{a}) \\ 1 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} L_0(\underline{a}', p) & \widetilde{L}_0(\underline{a}', p) \\ \omega^{s(p)}L_0(\omega.\underline{a}', p) & \omega^{s(-p)}\widetilde{L}_0(\omega.\underline{a}', p) \end{pmatrix}\begin{pmatrix} \omega^{s(p)}L_0(\omega.\underline{a}', p) & \omega^{s(-p)}\widetilde{L}_0(\omega.\underline{a}', p) \\ \omega^{2s(p)}L_0(\omega^2.\underline{a}', p) & \omega^{2s(-p)}\widetilde{L}_0(\omega^2.\underline{a}', p) \end{pmatrix}^{-1}$$

so that, with (59):

$$C_0(\underline{a}) = -\frac{p}{2}\omega^{1-r(\omega.\underline{a})}\left(\omega^{-p}\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} - \omega^{p}\frac{L_0(\omega^2.\underline{a}', p)}{\widetilde{L}_0(\omega^2.\underline{a}', p)}\right)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p). \qquad (69)$$

We now apply formula (67) under the assumptions made in theorem 6.5.

- When $m = 1$, $\omega = e^{2i\pi}$ and $r(\underline{a}) = \dfrac{1}{4}$, and formula (67) of theorem 6.5 reduces to:
$$\frac{L_0(p)}{\widetilde{L}_0(p)} = -\frac{1}{p\sin(\pi p)\widetilde{L}_0(p)\widetilde{L}_0(p)}.$$
This allows to write (69) as:
$$C_0(\underline{a}) = -\frac{\sin(2\pi p)}{\sin(\pi p)} = -2\cos(\pi p).$$
This result extends for all $\underline{a} \in \mathbb{C}$ by analytic continuation, since C_0 is entire in $\underline{a}$.

- When $m = 2$, we have $\omega = e^{i\pi}$ and $r(\underline{a}) = -\dfrac{a_1}{2}$.
Formula (67) of theorem 6.5 becomes:
$$\frac{L_0(\underline{a}', p)}{\widetilde{L}_0(\underline{a}', p)} = \frac{\omega^{-\frac{3p}{2}}}{p\sin(\pi p)}\left(\frac{\omega^{-\frac{a_1}{2}+p}}{\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)} + \frac{\omega^{\frac{a_1}{2}+2p}}{\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\underline{a}', p)}\right),$$

or also

$$\frac{L_0(\underline{a}',p)}{\widetilde{L}_0(\underline{a}',p)} = 2\frac{\cos\left(\frac{\pi}{2}(p+a_1)\right)}{p\sin(\pi p)} \frac{1}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)}.$$

This means that (69) reads:

$$C_0(\underline{a}) = -2ie^{-i\pi\frac{a_1}{2}}\cos\left(\frac{\pi}{2}(p+a_1)\right)\frac{\widetilde{L}_0(\underline{a}',p)}{\widetilde{L}_0(\omega.\underline{a}',p)}.$$

The announced result follows by analytic continuation for all $\underline{a}' \in \mathbb{C}$ and all $p \notin -\mathbb{N}$, since C_0 is entire in $\underline{a}$, while $\widetilde{L}_0$ is holomorphic in $\underline{a}' \in \mathbb{C}$ and $p \notin -\mathbb{N}$.

- When $m > 2$, we can write by theorem 6.5

$$C_0(\underline{a}) = i\frac{\omega^{-\frac{1}{2}}\omega^{-(m-1)\frac{p}{2}-r(\omega.\underline{a})}}{2\sin(\pi p)}\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega^2.\underline{a}',p)\times$$

$$\left(\omega^{-p}\sum_{l=0}^{m-1}\frac{\omega^{lp+r(\omega^l.\underline{a})}}{\widetilde{L}_0(\omega^l.\underline{a}',p)\widetilde{L}_0(\omega^{l+1}.\underline{a}',p)} - \omega^p\sum_{l=0}^{m-1}\frac{\omega^{lp+r(\omega^{l+2}.\underline{a})}}{\widetilde{L}_0(\omega^{l+2}.\underline{a})\widetilde{L}_0(\omega^{l+3}.\underline{a}',p)}\right),$$

which reads also:

$$C_0(\underline{a}) =$$

$$i\frac{\omega^{-\frac{1}{2}}\omega^{-(m-1)\frac{p}{2}-r(\omega.\underline{a})}}{2\sin(\pi p)}\left(\frac{\omega^{-p+r(\underline{a})}}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)} + \frac{\omega^{r(\omega.\underline{a})}}{\widetilde{L}_0(\omega.\underline{a}',p)\widetilde{L}_0(\omega^2.\underline{a}',p)} +\right.$$

$$\sum_{l=2}^{m-1}\frac{\omega^{(l-1)p+r(\omega^l.\underline{a})}}{\widetilde{L}_0(\omega^l.\underline{a}',p)\widetilde{L}_0(\omega^{l+1}.\underline{a}',p)} - \sum_{l=0}^{m-3}\frac{\omega^{(l+1)p+r(\omega^{l+2}.\underline{a})}}{\widetilde{L}_0(\omega^{l+2}.\underline{a}',p)\widetilde{L}_0(\omega^{l+3}.\underline{a}',p)}$$

$$\left.-\frac{\omega^{(m-1)p+r(\underline{a})}}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)} - \frac{\omega^{mp+r(\omega.\underline{a})}}{\widetilde{L}_0(\omega.\underline{a}',p)\widetilde{L}_0(\omega^2.\underline{a}',p)}\right)\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega^2.\underline{a}',p).$$

The right-hand side of this equality simplifies to give

$$C_0(\underline{a}) =$$

$$\omega^{r(\underline{a})-1+\frac{m}{4}}\left(\frac{\widetilde{L}_0(\underline{a}',p)}{\widetilde{L}_0(\omega.\underline{a}',p)}\omega^{-r(\underline{a})+\frac{1}{2}-\frac{m}{4}+\frac{p}{2}} + \frac{\widetilde{L}_0(\omega^2.\underline{a}',p)}{\widetilde{L}_0(\omega.\underline{a}',p)}\omega^{r(\underline{a})-\frac{1}{2}+\frac{m}{4}-\frac{p}{2}}\right).$$

Again, the announced result follows by analytic continuation.

$\square$

6.3 Third functional equation

In this subsection, we study a class of differential equations $(\mathfrak{E}_m)$ with higher symmetries. For that purpose, it will be useful to introduce new notations.

Notation 6.7. For $m, n \in \mathbb{N}^\star$, we define

$$\underline{a}_n = (0, \cdots, 0, a_n, 0, \cdots, 0, a_{2n}, 0, \cdots, 0, a_{jn}, 0 \cdots, 0, a_{nm}) \in \mathbb{C}^{nm},$$

i.e.,

$$\underline{a}_n = (a_j)_{1 \le j \le nm} \quad \text{so that} \quad a_j = 0 \quad \text{if} \quad j \ne 0 \mod m.$$

For such a $\underline{a}_n$, we also define:

$$\underline{a}_n' := (a_j)_{1 \le j \le nm-1}$$

and

$$\begin{cases} \underline{\widetilde{a}}_n := \left(\dfrac{a_n}{n^{\frac{2}{m}}}, \dfrac{a_{2n}}{n^{\frac{4}{m}}}, \cdots, \dfrac{a_{n(m-1)}}{n^{\frac{2(m-1)}{m}}}, -\dfrac{1}{4} + \dfrac{1 + 4a_{nm}}{4n^2} \right) \in \mathbb{C}^m \\[2ex] \underline{\widetilde{a}}_n' := \left(\dfrac{a_n}{n^{\frac{2}{m}}}, \dfrac{a_{2n}}{n^{\frac{4}{m}}}, \cdots, \dfrac{a_{n(m-1)}}{n^{\frac{2(m-1)}{m}}} \right) \in \mathbb{C}^{m-1}. \end{cases}$$

We shall consider in this subsection the following differential equation:

$$(\mathfrak{E}_{nm}^n) \qquad x^2 \frac{d^2}{dx^2} \Phi(x, \underline{a}_n) = P_{nm}(x, \underline{a}_n) \Phi(x, \underline{a}_n).$$

This equation is a particular case of our main equation $(\mathfrak{E}_{nm})$, but its higher symmetry will allow us to compare its Stokes-Sibuya and 0∞ connection matrices with those of $(\mathfrak{E}_m)$, associated with the polynomial $P_m(x, \underline{\widetilde{a}}_n)$ of lower order.

We begin with a lemma:

Lemma 6.8. *If Φ satisfies the differential equation $(\mathfrak{E}_{nm}^n)$ with $n, m \in \mathbb{N}^\star$, then Ψ defined by*

$$\Psi(x, \underline{\widetilde{a}}_n) := x^{\frac{n-1}{2n}} \Phi\left((n^{\frac{2}{m}} x)^{\frac{1}{n}}, \underline{a}_n \right)$$

satifies the differential equation $(\mathfrak{E}_m)$ with $\underline{a} = \underline{\widetilde{a}}_n$, that is:

$$x^2 \frac{d^2}{dx^2} \Psi(x, \underline{\widetilde{a}}_n) = P_m(x, \underline{\widetilde{a}}_n) \Psi(x, \underline{\widetilde{a}}_n). \tag{70}$$

Proof. We consider the transformation

$$\Psi(x, \underline{\widetilde{a}}_n) = x^\alpha \Phi(\lambda x^{\frac{1}{n}}, \underline{a}_n)$$

with $\alpha = \dfrac{n-1}{2n}$. Then

$$x^2 \Psi''(x, \underline{\widetilde{a}}_n) = \frac{\lambda^2}{n^2} x^{\alpha + \frac{2}{n}} \Phi''(\lambda x^{\frac{1}{n}}, \underline{a}_n) + \alpha(\alpha - 1) x^\alpha \Phi(\lambda x^{\frac{1}{n}}, \underline{a}_n).$$

Assuming that $x^2 \Phi''(x, \underline{a}_n) = P_{nm}(x, \underline{a}_n) \Phi(x, \underline{a}_n)$, one gets

$$x^2 \Psi''(x, \underline{\widetilde{a}}_n) = \left(\frac{1}{n^2} P_{nm}(\lambda x^{\frac{1}{n}}, \underline{a}_n) + \alpha(\alpha - 1) \right) \Psi(x, \underline{\widetilde{a}}_n).$$

We choose $\lambda = n^{\frac{2}{nm}}$ to get the statement. ∎

Notation 6.9. We note $C_k^n(\underline{a}_n)$ and $\widetilde{C}_k^n(\underline{a}_n)$, $k \in \mathbb{Z}$, the Stokes-Sibuya coefficients associated with equation $(\mathfrak{E}_{nm}^n)$.

The above lemma induces the following corollary:

Corollary 6.10. *The Stokes-Sibuya coefficients $C_0^n(\underline{a}_n)$ and $\widetilde{C}_0^n(\underline{a}_n)$ associated with equation $(\mathfrak{E}_{nm}^n)$ are related to the Stokes-Sibuya coefficients C_0 and $\widetilde{C}_0$ of equation $(\mathfrak{E}_m)$ by:*

$$\begin{aligned} C_0^n(\underline{a}_n) &= \omega^{\frac{n-1}{2n}} C_0\left(\underline{\widetilde{a}}_n\right) \\ \widetilde{C}_0^n(\underline{a}_n) &= \omega^{\frac{n-1}{n}} \widetilde{C}_0\left(\underline{\widetilde{a}}_n\right) \end{aligned} \tag{71}$$

where $\omega = e^{\frac{2i\pi}{m}}$.

Proof. We note Φ_0 the solution of $(\mathfrak{E}_m)$ which is characterized by its asymptotics

$$T\Phi_0(x, \underline{a}) = x^{r_m(\underline{a})} e^{-S_m(x,\underline{a})}\left(1 + o(1)\right)$$

at infinity in the sector $\Sigma_0 = \{|x| > 0, |\arg(x)| < \frac{3\pi}{m}\}$ (where $r_m = r$ and $S_m = S$ in theorem 2.1). The Stokes-Sibuya coefficients C_0 and $\widetilde{C}_0$ are defined by:

$$\begin{aligned} &\Phi_0(\omega^{-1}x, \omega^{-1}.\underline{a}) = \\ &C_0(\underline{a})\Phi_0(x, \underline{a}) + \widetilde{C}_0(\underline{a})\Phi_0(\omega x, \omega.\underline{a}) \end{aligned} \tag{72}$$

with $\omega = e^{\frac{2i\pi}{m}}$. We note Φ_0^n its analog for equation $(\mathfrak{E}_{nm}^n)$, so that

$$T\Phi_0^n(x, \underline{a}_n) = x^{r_{nm}(\underline{a}_n)} e^{-S_{nm}(x,\underline{a}_n)}\left(1 + o(1)\right)$$

at infinity in the sector $\Sigma_0^n = \{|x| > 0, |\arg(x)| < \frac{3\pi}{nm}\}$ (where $r_{nm} = r$ and $S_{nm} = S$ in theorem 2.1), and

$$\Phi_0^n(\omega_n^{-1}x, \omega_n^{-1}.\underline{a}_n) = C_0^n(\underline{a}_n)\Phi_0^n(x, \underline{a}_n) + \widetilde{C}_0^n(\underline{a}_n)\Phi_0^n(\omega_n x; \omega_n.\underline{a}_n) \tag{73}$$

with $\omega_n = e^{\frac{2i\pi}{mn}}$. Introducing, with lemma 6.8, the function

$$\Psi_0(x, \underline{\widetilde{a}}_n) = x^{\frac{n-1}{2n}} \Phi_0^n\left((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}_n\right) \tag{74}$$

we get a solution of $(\mathfrak{E}_m)$ such that

$$T\Psi_0(x, \underline{\widetilde{a}}_n) = n^{\frac{2}{nm}r_{nm}(\underline{a}_n)} x^{\frac{1}{n}r_{nm}(\underline{a}_n) + \frac{n-1}{2n}} e^{-S_{nm}((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}_n)}\left(1 + o(1)\right)$$

at infinity in the sector Σ_0. One easily checks that $S_{nm}((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}_n) = S_m(x, \underline{\widetilde{a}}_n)$ and $\dfrac{1}{n}r_{nm}(\underline{a}_n) = r_m(\underline{\widetilde{a}}_n) - \dfrac{n-1}{2n}$. This means that

$$\Psi_0(x, \underline{\widetilde{a}}_n) = n^{\frac{2}{m}r_m(\underline{\widetilde{a}}_n) - \frac{n-1}{mn}} \Phi_0\left(x, \underline{\widetilde{a}}_n\right). \tag{75}$$

From (73), one observes that

$$\begin{aligned} &\omega^{\frac{n-1}{2n}}\left(\omega^{-1}x\right)^{\frac{n-1}{2n}} \Phi_0^n((n^{\frac{2}{m}}\omega^{-1}x)^{\frac{1}{n}}, \omega_n^{-1}.\underline{a}_n) = \\ &C_0^n(\underline{a}_n)x^{\frac{n-1}{2n}}\Phi_0^n((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}_n) + \omega^{-\frac{n-1}{2n}}\widetilde{C}_0^n(\underline{a}_n)\left(\omega x\right)^{\frac{n-1}{2n}} \Phi_0^n((n^{\frac{2}{m}}\omega x)^{\frac{1}{n}}; \omega_n.\underline{a}_n) \end{aligned} \tag{76}$$

so that, by (74),

$$\begin{gathered}\omega^{\frac{n-1}{2n}}\Psi_0(\omega^{-1}x,\omega^{-1}.\underline{\widetilde{a}}_n) = \\ C_0^n(\underline{a}_n)\Psi_0(x,\underline{\widetilde{a}}_n) + \omega^{-\frac{n-1}{2n}}\widetilde{C}_0^n(\underline{a}_n)\Psi_0(\omega x,\omega.\underline{\widetilde{a}}_n).\end{gathered} \tag{77}$$

Using (75), one can compare this last equation with (72) to get the final result. □

Lemma 6.8 can be used also to compare the 0∞ connection matrices. We shall use the following notations:

Notation 6.11. We note $\widetilde{L}_k^n(\underline{a}'_n, p(a_{mn}))$ and $L_k^n(\underline{a}'_n, p(a_{mn}))$, $k \in \mathbb{Z}$, the coefficients of the 0∞ connection matrices associated with equation $(\mathfrak{E}_{nm}^n)$ with $p(a_{mn}) = (1+4a_{mn})^{\frac{1}{2}}$.

Corollary 6.12. *When* $-\dfrac{p(a_{mn})}{n} \notin \mathbb{N}$,

$$\widetilde{L}_0^n(\underline{a}'_n, p(a_{mn})) = e^{\frac{i\pi}{m}(1-\frac{1}{n})} n^{-\frac{2}{m}r(\underline{\widetilde{a}}_n)+\frac{p(a_{mn})}{mn}+\frac{1}{m}-1}\widetilde{L}_0\left(\underline{\widetilde{a}}'_n, \frac{p(a_{mn})}{n}\right). \tag{78}$$

Proof. We use the notations of theorem 4.2. We introduce the solution $f_1(x,\underline{a})$ of $(\mathfrak{E}_m)$ which reads

$$f_1(x,\underline{a}',p(a_m)) = x^{s(p(a_m))}g_1(x,\underline{a}',p(a_m)), \tag{79}$$

where we assume that $p(a_m) = (1+4a_m)^{\frac{1}{2}} \notin -\mathbb{N}$. In the same way, we note $f_1^n(x,\underline{a}'_n,p(a_{mn}))$ the solution of $(\mathfrak{E}_{nm}^n)$ which can be written as

$$f_1^n(x,\underline{a}'_n,p(a_{mn})) = x^{s(p(a_{mn}))}g^n(x,\underline{a}'_n,p(a_{mn})) \tag{80}$$

under the condition $p(a_{mn}) \notin -\mathbb{N}$. Following lemma 6.8, we define

$$F_1(x,\underline{\widetilde{a}}_n) = x^{\frac{n-1}{2n}} f_1^n\left((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}'_n, p(a_{mn})\right) \tag{81}$$

which is solution of $(\mathfrak{E}_m)$ with parameter $\underline{\widetilde{a}}_n$. One easily checks that, necessarily,

$$F_1(x,\underline{\widetilde{a}}_n) = n^{\frac{2}{mn}s(p(a_{mn}))} f_1\left(x, \underline{\widetilde{a}}'_n, \frac{p(a_{mn})}{n}\right). \tag{82}$$

In other words,

$$f_1\left(x,\underline{\widetilde{a}}'_n,\frac{p(a_{mn})}{n}\right) = n^{-\frac{2}{mn}s(p(a_{mn}))}x^{\frac{n-1}{2n}} f_1^n\left((n^{\frac{2}{m}}x)^{\frac{1}{n}}, \underline{a}'_n, p(a_{mn})\right). \tag{83}$$

Note that this equality allows to extends analytically $f_1^n(x,\underline{a}'_n,p(a_{mn}))$ for $\dfrac{p(a_{mn})}{n} \notin -\mathbb{N}$, and this translates to the $\widetilde{L}_k^n(\underline{a}'_n,p(a_{mn}))$ as well.

We consider the 0∞ connection matrices M_1 and M_1^n associated with $(\mathfrak{E}_m)$ and $(\mathfrak{E}_{nm}^n)$ respectively. We have

$$\begin{gathered}\widetilde{L}_1(\underline{a}',p(a_m)) = \\ -\tfrac{1}{p(a_m)}\left(f_1(x,\underline{a}',p(a_m))\Phi_0'(x,\underline{a}) - f_1'(x,\underline{a}',p(a_m))\Phi_0(x,\underline{a})\right)\end{gathered} \tag{84}$$

and

$$\widetilde{L}_1^n(\underline{a}'_n, p(a_{mn})) = \\ -\tfrac{1}{p(a_{mn})}\left(f_1^n(x,\underline{a}'_n,p(a_{mn}))\Phi_0^{n\prime}(x,\underline{a}_n) - f_1^{n\prime}(x,\underline{a}'_n,p(a_{mn}))\Phi_0^n(x,\underline{a}_n)\right) \tag{85}$$

where Φ_0^n has been defined in the proof of corollary 6.10. By (74) and (75), we know that

$$\Phi_0\left(x,\underline{\widetilde{a}}_n\right) = n^{-\frac{2}{m}r(\underline{\widetilde{a}}_n)+\frac{n-1}{mn}}x^{\frac{n-1}{2n}}\Phi_0^n\left((n^{\frac{2}{m}}x)^{\frac{1}{n}},\underline{a}_n\right). \tag{86}$$

Equation (84) together with (83) and (86) yields:

$$\widetilde{L}_1\left(\underline{\widetilde{a}}'_n, \tfrac{p(a_{mn})}{n}\right) = -n^{-\frac{2}{m}r(\underline{\widetilde{a}}_n)-\frac{p(a_{mn})}{mn}+\frac{1}{m}}\times$$

$$\tfrac{1}{p(a_{mn})}\left(f_1^n((n^{\frac{2}{m}}x)^{\frac{1}{n}},\underline{a}'_n,p(a_{mn}))\Phi_0^{n\prime}((n^{\frac{2}{m}}x)^{\frac{1}{n}},\underline{a}_n)-\right. \tag{87}$$

$$\left. f_1^{n\prime}((n^{\frac{2}{m}}x)^{\frac{1}{n}},\underline{a}'_n,p(a_{mn}))\Phi_0^n((n^{\frac{2}{m}}x)^{\frac{1}{n}},\underline{a}_n)\right).$$

Comparing (87) with (85), we get

$$\widetilde{L}_1^n(\underline{a}'_n, p(a_{mn})) = n^{\frac{2}{m}r(\underline{\widetilde{a}}_n)+\frac{p(a_{mn})}{mn}-\frac{1}{m}}\widetilde{L}_1\left(\underline{\widetilde{a}}'_n, \frac{p(a_{mn})}{n}\right). \tag{88}$$

Using formula (61) of theorem 5.3, we eventually find, since $r(\underline{a})+r(\omega.\underline{a}) = 1-\dfrac{m}{2}$,

$$\widetilde{L}_0^n(\underline{a}'_n, p(a_{mn})) = e^{\frac{i\pi}{m}(1-\frac{1}{n})}n^{-\frac{2}{m}r(\underline{\widetilde{a}}_n)+\frac{p(a_{mn})}{mn}+\frac{1}{m}-1}\widetilde{L}_0\left(\underline{\widetilde{a}}'_n, \frac{p(a_{mn})}{n}\right). \tag{89}$$

□

7 Some applications

7.1 Application for a special class

Some simplifications occur when $\underline{a}'=0$, allowing to get the following proposition:

Proposition 7.1. *We consider $(\mathfrak{E}_m)$ on restriction to $\underline{a}'=0$. Then*

$$\mathfrak{S}_0(0,a_m) = \begin{pmatrix} 2e^{-\frac{i\pi}{m}}\cos\left(\pi\dfrac{p}{m}\right) & -e^{-\frac{2i\pi}{m}} \\ 1 & 0 \end{pmatrix}$$

where $p=(1+4a_m)^{1/2}$. Furthermore, for $\dfrac{p}{m}\notin\mathbb{Z}$, the 0∞ connection matrix M_0 is given by

$$M_0(0,p) = \begin{pmatrix} e^{\beta_m(-p)}\dfrac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(-\dfrac{p}{m}\right) & e^{\beta_m(p)}\dfrac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(\dfrac{p}{m}\right) \\ \omega^{s(p)}e^{\beta_m(-p)}\dfrac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(-\dfrac{p}{m}\right) & \omega^{s(-p)}e^{\beta_m(p)}\dfrac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(\dfrac{p}{m}\right) \end{pmatrix}$$

where $s(p)=\dfrac{1+p}{2}$, while $\beta_m(p)$ is an odd function, entire in p, such that for all $k\ \in\mathbb{N}^\star$,

$$e^{\beta_m(km)} = \pm m^k.$$

Remark 7.2. We shall see in a moment (§A.1) by other means that $\widetilde{L}_0(\underline{a}) = -e^{i\pi p}\dfrac{\Gamma(p)}{\sqrt{\pi}}$ when $m = 1$. Moreover $r(\underline{a}) = \dfrac{1}{4}$ for $m = 1$. Applying corollary 6.12 with $\underline{a}' = 0$, we deduce that,

$$\widetilde{L}_0(0,p) = e^{-\frac{i\pi}{m}} m^{\frac{p}{m}} e^{i\pi\frac{p}{m}} \frac{\Gamma(\frac{p}{m})}{\sqrt{m\pi}}$$

while

$$L_0(0,p) = e^{-\frac{i\pi}{m}} m^{-\frac{p}{m}} e^{-i\pi\frac{p}{m}} \frac{\Gamma(-\frac{p}{m})}{\sqrt{m\pi}}.$$

Proof. We note that when $\underline{a}' = 0$, then $\omega.\underline{a} = \underline{a}$ and $r(\underline{a}) = \dfrac{1}{2} - \dfrac{m}{4}$. This has two consequences.

Firstly, when applying corollary 6.6, one immediately gets, for $p \notin -\mathbb{N}$,

$$C_0(0, a_m) = 2e^{-\frac{i\pi}{m}} \cos\left(\pi\frac{p}{m}\right).$$

Since $C_0(\underline{a})$ is an entire function in $\underline{a}$, the above formula extends to all a_m, by analytic continuation.

Secondly, formula (67) of theorem 6.5 reduces into

$$\frac{1}{L_0(0,p)\widetilde{L}_0(0,p)} = -\omega p \sin\left(\pi\frac{p}{m}\right),$$

which resembles the Euler reflection formula $\dfrac{\pi}{\Gamma(z)\Gamma(-z)} = -z\sin(\pi z)$. Since by proposition 5.4, the restriction to $p \notin \mathbb{Z}$ of the function $L_0(0,p)$ (*resp.* $\widetilde{L}_0(0,p)$) has a meromorphic continuation in p, with at most simple poles at $p \in m\mathbb{N}$ (*resp.* $-p \in m\mathbb{N}$), we can write

$$L_0(0,p) = \alpha(p)\frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(-\frac{p}{m}\right)$$

and

$$\widetilde{L}_0(0,p) = \frac{1}{\alpha(p)}\frac{\omega^{-\frac{1}{2}}}{\sqrt{m\pi}}\Gamma\left(\frac{p}{m}\right)$$

where $\alpha(p)$ is a nowhere vanishing entire function of p. In other words,

$$\alpha(p) = e^{-\beta_m(p)}$$

with $\beta_m(p)$ an entire function. Furthermore, by theorem 5.3 again, we know that $L_0(\underline{a}', -p) = \widetilde{L}_0(\underline{a}', p)$. This means that β_m can be chosen as an odd function.

Still by theorem 5.3, we know that $\widetilde{L}_0(\underline{a}', p)$ extends analytically to $p \in \mathbb{N}^\star$. Moreover, when

$$p = km, \quad k \in \mathbb{N}^\star,$$

then formula (68) of theorem 6.5 gives

$$\widetilde{L}_0^2(0,p)\,|_{p=km} = (-1)^{k+1}\frac{m\omega^{-1}}{\pi p\lambda(0,p)}\,|_{p=km}.$$

By remark 4.6, we know that

$$\lambda(0,p)\,|_{p=km} = \frac{(-1)^{k+1}}{m^{2k-1}k\Gamma(k)^2},$$

and therefore,

$$\widetilde{L}_0^2(0,p)\,|_{p=km} = m^{2k}\frac{\omega^{-1}\Gamma^2(k)}{m\pi},$$

that is

$$\widetilde{L}_0(0,p)\,|_{p=km} = \pm m^k\frac{\omega^{-1/2}\Gamma(k)}{\sqrt{m\pi}}.$$

□

7.2 Application when $m=2$ and consequences.

We consider the case $m=2$, so that $\omega = e^{i\pi}$ and $r(\underline{a}) = -\frac{a_1}{2}$.

On the one hand, corollary 6.3 implies

$$C_0(\underline{a})C_1(\underline{a}) + \widetilde{C}_0(\underline{a}) + \widetilde{C}_1(\underline{a}) = -2\cos(\pi p)$$

with $C_1(\underline{a}) = C_0(\omega.\underline{a})$, where, by (7) of theorem 2.7,

$$\widetilde{C}_0(\underline{a}) = e^{-i\pi a_1}, \qquad \widetilde{C}_1(a) = \widetilde{C}_0(\omega.\underline{a}) = e^{i\pi a_1}.$$

This means that

$$C_0(\underline{a})C_0(\omega.\underline{a}) = -4\cos\left(\frac{\pi}{2}(p+a_1)\right)\cos\left(\frac{\pi}{2}(p-a_1)\right).$$

On the other hand, we know by corollary 6.6 that (for $\underline{a}$ generic):

$$C_0(\underline{a}) = -2ie^{-i\pi\frac{a_1}{2}}\cos\left(\frac{\pi}{2}(p+a_1)\right)\frac{\widetilde{L}_0(\underline{a}',p)}{\widetilde{L}_0(\omega.\underline{a}',p)}. \tag{90}$$

Also, by formula (68) of theorem 6.5, we have, when $p \in \mathbb{N}^\star$ and for $\underline{a}' = a_1$ generic,

$$\pi p\lambda(\underline{a}',p) = 2\frac{\cos\left(\frac{\pi}{2}(p-a_1)\right)}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)}\,|_{p\in\mathbb{N}^\star}.$$

By remark 4.6, $\begin{cases} \lambda(\underline{a}',p)\,|_{p=0 \mod 2} = \dfrac{(-1)^{p+1}}{p\Gamma(p)^2}\displaystyle\prod_{k=1}^{p/2}(a_1+2k-1)(a_1-2k+1) \\ \\ \lambda(\underline{a}',p)\,|_{p=1 \mod 2} = \dfrac{(-1)^{p+1}}{p\Gamma(p)^2}a_1\displaystyle\prod_{k=1}^{(p-1)/2}(a_1+2k)(a_1-2k), \end{cases}$ so that

$$\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)\,|_{p\in\mathbb{N}^\star} =$$

$$\begin{cases} 2(-1)^{\frac{p+2}{2}} \dfrac{\cos\left(\frac{\pi}{2}a_1\right)\Gamma(p)^2}{\pi\prod_{k=1}^{p/2}(a_1+2k-1)(a_1-2k+1)}, & p \text{ even} \\ 2(-1)^{\frac{p+1}{2}} \dfrac{\sin\left(\frac{\pi}{2}a_1\right)\Gamma(p)^2}{\pi a_1\prod_{k=1}^{(p-1)/2}(a_1+2k)(a_1-2k)}, & p \text{ odd.} \end{cases}$$

This can be also written as

$$\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)\,|_{p\in\mathbb{N}^\star} = -2^{-p+1}e^{i\pi p}\frac{\Gamma(p)^2}{\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})}. \tag{91}$$

At this point, we can use the following lemma, whose easy proof is left to the reader:

Lemma 7.3. *When $p+a_1+1=-2N$ with $N\in\mathbb{N}$, then, for $p\notin -\mathbb{N}^\star$,*

$$f_1(x,a_1,p) = x^{s(p)}e^{-x}\sum_{n=0}^{N}\frac{\Gamma(p+1)Q_n(a_1,p)}{\Gamma(n+p+1)}x^n$$

where the $Q_n(a_1,p)\in\mathbb{C}[a_1,p]$ are defined by:

$$\begin{cases} Q_0(a_1,p)=1 \\ (a_1+p+1+2n)Q_n(a_1,p)-(n+1)Q_{n+1}(a_1,p)=0, \ n\geq 0. \end{cases}$$

In particular, $f_1(x,a_1,p) = (-1)^N 2^N \dfrac{\Gamma(p+1)}{\Gamma(N+p+1)}\Phi_0(x,\underline{a})$.

This lemma implies that

$$\widetilde{L}_0(\omega.\underline{a}',p)=0 \quad \text{when} \quad \begin{cases} p+a_1+1\in -2\mathbb{N} \\ p\notin -\mathbb{N}^\star \end{cases},$$

that is also

$$\widetilde{L}_0(\underline{a}',p)=0 \quad \text{when} \quad \begin{cases} p-a_1+1\in -2\mathbb{N} \\ p\notin -\mathbb{N}^\star. \end{cases} \tag{92}$$

Since the right-hand side of (91) has only simple zeros when $p+a_1+1\in -2\mathbb{N}$, we can write:

$$\begin{aligned} &\widetilde{L}_0(\underline{a}',p)\,|_{p\in\mathbb{N}^\star} = -i2^{-\frac{p-1}{2}}e^{i\pi\frac{p}{2}}\frac{\Gamma(p)}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})}e^{\beta(a_1,p)}\,|_{p\in\mathbb{N}^\star} \\ &\text{with} \quad \beta(-a_1,p)=-\beta(a_1,p). \end{aligned} \tag{93}$$

Now when $p\notin\mathbb{Z}$, the coefficient $L_0(\underline{a}',p)$ can be derived from formula (67) of theorem 6.5. This gives:

$$L_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p) = 2\frac{\cos\left(\frac{\pi}{2}(p+a_1)\right)}{p\sin(\pi p)}. \tag{94}$$

We recall also that $L_0(\underline{a}',p)$ can be derived from $\widetilde{L}_0(\underline{a}',p)$ just by changing p into $-p$. Using (92) and the known analytic properties of $L_0(\underline{a}',p)$ and $\widetilde{L}_0(\underline{a}',p)$ described by proposition 5.4, equation (94) shows that equation (93) can be extended to all $(\underline{a}',p)\in\mathbb{C}^2$ with $p\notin -\mathbb{N}^\star$,

$$\widetilde{L}_0(\underline{a}',p) = -i2^{-\frac{p-1}{2}}e^{i\pi\frac{p}{2}}\frac{\Gamma(p)}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})}e^{\beta(a_1,p)}$$

where β can be chosen as an entire function satisfying

$$\beta(-a_1,p) = -\beta(a_1,p) \quad \text{and} \quad \beta(a_1,-p) = \beta(a_1,p).$$

Finally, formula 90 reduces into:

$$C_0(\underline{a}) = -2ie^{-i\pi\frac{a_1}{2}} \cos\left(\frac{\pi}{2}(p+a_1)\right) \frac{\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} e^{2\beta(a_1,p)}.$$

To summarize:

Proposition 7.4. *We assume $m=2$. Then the Stokes-Sibuya mutiplier C_0 may be written as*

$$C_0(\underline{a}) = -2ie^{-i\pi\frac{a_1}{2}} \cos\left(\frac{\pi}{2}(p+a_1)\right) \frac{\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} e^{2\beta(a_1,p)} \quad \textit{and} \quad \widetilde{C}_0(\underline{a}) = e^{-i\pi a_1} \tag{95}$$

where β is an entire function satisfying $\beta(a_1,p) = \beta(a_1,-p) = -\beta(-a_1,p)$.
Moreover, the coefficients of the 0∞ connection matrix M_0 of theorem 5.3 satisfy, for $p \notin \mathbb{Z}$,

$$\begin{cases} \widetilde{L}_0(a_1,p) = -i2^{-\frac{p-1}{2}} e^{i\pi\frac{p}{2}} \dfrac{\Gamma(p)}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} e^{\beta(a_1,p)} \\ \\ L_0(a_1,p)\widetilde{L}_0(\omega a_1,p) = 2\dfrac{\cos\left(\frac{\pi}{2}(p+a_1)\right)}{p\sin(\pi p)}. \end{cases} \tag{96}$$

Remark 7.5. The above proposition is interesting since, for instance, it already provides the location of the zeroes of C_0 and of the other Stokes-Sibuya coefficients. However, one can be more precise, using the Whittaker special functions. We shall see in a moment (§A.2) that

$$\beta(a_1,p) = -a_1.$$

With this remark and corollaries 6.10 and 6.12, proposition 7.4 implies the following consequences:

Corollary 7.6. *We consider the differential equation*

$$(\mathfrak{E}^n_{2n}) \qquad x^2\frac{d^2}{dx^2}\Phi = \left(x^{2n} + a_n x^n + a_{2n}\right)\Phi$$

where $n \in \mathbb{N}^\star$. Then,

$$\begin{cases} C^n_0(\underline{a}_n) = 2e^{-\frac{i\pi}{2n}} e^{-i\pi\frac{a_n}{2n}} 2^{-\frac{a_n}{n}} \dfrac{\Gamma(\frac{p}{2n}+\frac{a_n}{2n}+\frac{1}{2})}{\Gamma(\frac{p}{2n}-\frac{a_n}{2n}+\frac{1}{2})} \cos\left((\dfrac{p}{2n}+\dfrac{a_n}{2n})\pi\right) \\ \\ \widetilde{C}^n_0(\underline{a}_n) = -e^{-\frac{i\pi}{n}} e^{-i\pi\frac{a_n}{n}} \end{cases}$$

where $p = (1+4a_{2n})^{\frac{1}{2}}$. Moreover, when $p \notin -n\mathbb{N}$,

$$\widetilde{L}^n_0(\underline{a}'_n,p) = e^{-\frac{i\pi}{2n}} e^{i\pi\frac{p}{2n}} \left(\frac{n}{2}\right)^{\frac{a_n}{2n}+\frac{p}{2n}-\frac{1}{2}} \frac{\Gamma(\frac{p}{n})}{\Gamma(\frac{p}{2n}-\frac{a_n}{2n}+\frac{1}{2})}.$$

7.3 Application when $m \geq 3$.

As a matter of fact, we shall only discuss the cases $m = 3$ and $m = 4$ to show what kind of information we can extract from our analysis.

7.3.1 The case $m = 3$.

In a sense, $m = 3$ is the first interesting case, since no special function solution of $(\mathfrak{E}_3)$ is known.
Here we have $\omega = e^{\frac{2i\pi}{3}}$ and $r(\underline{a}) = -\dfrac{1}{4}$ is a constant function.

We first apply corollary 6.3, to get a functional relation between the Stokes-Sibuya multipliers:

$$C_0(\underline{a})C_1(\underline{a})C_2(\underline{a}) + \widetilde{C}_0(\underline{a})C_2(\underline{a}) + \widetilde{C}_1(\underline{a})C_0(\underline{a}) + \widetilde{C}_2(\underline{a})C_1(\underline{a}) = -2\cos(\pi p)$$

where, by (7) of theorem 2.7,

$$\widetilde{C}_0(\underline{a}) = \widetilde{C}_1(\underline{a}) = \widetilde{C}_2(\underline{a}) = e^{\frac{i\pi}{3}}.$$

Applying now corollary 6.6, we find, for all $\underline{a}' \in \mathbb{C}^2$ and $p \notin -\mathbb{N}$:

$$\widetilde{L}_0(\omega.\underline{a}', p)C_0(\underline{a}) = \omega^{-\frac{1}{2}}\left(\widetilde{L}_0(\underline{a}', p)\omega^{\frac{p}{2}} + \widetilde{L}_0(\omega^2.\underline{a}', p)\omega^{-\frac{p}{2}}\right). \tag{97}$$

We concentrate on the case $p \in \mathbb{N}^\star$. By formula (68) of theorem 6.5, we get

$$\begin{gathered} i\pi p\omega^{\frac{7}{4}+\frac{p}{2}}\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = \\ \widetilde{L}_0(\omega.\underline{a}', p) + \omega^p\widetilde{L}_0(\omega^2.\underline{a}', p) + \omega^{2p}\widetilde{L}_0(\underline{a}', p). \end{gathered} \tag{98}$$

We now add the assumption that $\underline{a}'$ has been chosen so that

$$\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = 0.$$

Using the remark that $\widetilde{L}_0(\omega.\underline{a}', p) = 0$ implies $\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) \neq 0$ necessarily (otherwise, one of the two 0∞ connection matrices M_0 or M_1 is not invertible, which is absurd), equations (98) and (97) imply that:

$$C_0(\underline{a}) = C_1(\underline{a}) = C_2(\underline{a}) = -\omega^{-\frac{1}{2}-\frac{3p}{2}} = (-1)^{p+1}e^{-\frac{i\pi}{3}}.$$

We summarize our results:

Proposition 7.7. *We assume $m = 3$. Then the Stokes-Sibuya multiplier $C_0(\underline{a})$ satisfies the functional equation*

$$C_0(\underline{a})C_0(\omega.\underline{a})C_0(\omega^2.\underline{a}) + e^{\frac{i\pi}{3}}\left(C_0(\underline{a}) + C_0(\omega.\underline{a}) + C_0(\omega^2.\underline{a})\right) = -2\cos(\pi p) \tag{99}$$

with $p = (1 + 4a_3)^{\frac{1}{2}}$ and $\omega = e^{\frac{2i\pi}{3}}$, whereas

$$\widetilde{C}_0(\underline{a}) = e^{\frac{i\pi}{3}}. \tag{100}$$

Moreover, when $a_3 = \dfrac{p^2-1}{4}$ *with* $p \in \mathbb{N}^\star$*, then*

$$\lambda(\underline{a}', p)\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) = 0$$

is equivalent to C_0 *being a constant, precisely*

$$C_0 = (-1)^{p+1}e^{-\frac{i\pi}{3}}.$$

We note that proposition 7.7 can be derived from corollary 6.4 when $\lambda(\underline{a}', p) = 0$, while the particular case $a_1 = a_2 = 0$ is given by proposition 7.1.

For a given $p \in \mathbb{N}^\star$, the case $\lambda(\underline{a}', p) = 0$ can be seen as an isomonodromic deformation condition, since both the monodromy at the origin and the Stokes structure are fixed. We get:

Corollary 7.8. *For* $m = 3$ *and* $p \in \mathbb{N}^\star$*, the condition* $\lambda(\underline{a}', p) = 0$ *is an isomonodromic deformation condition.*

By computing $\lambda(\underline{a}', p)$ (see proposition 4.5), one obtains for example, from proposition 7.7:

- If $p = 1$, then $\lambda(\underline{a}', p) = a_2$. Therefore, for all $a_1 \in \mathbb{C}$,

$$C_0(a_1, 0, 0) = e^{-\frac{i\pi}{3}}.$$

  By a Tschirnhaus transformation, this case is equivalent to the Airy equation. This means also that $\widetilde{L}_0(a_1, 0, 1) = \widetilde{L}_0(0, 0, 1)$ so that by remark 7.2,

$$\widetilde{L}_0(a_1, 0, 1) = e^{-\frac{i\pi}{3}} 3^{\frac{1}{3}} e^{i\pi\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\sqrt{3\pi}},$$

  while $L_0(a_1, 0, 1) = e^{-\frac{i\pi}{3}} 3^{-\frac{1}{3}} e^{-i\pi\frac{1}{3}} \dfrac{\Gamma(-\frac{1}{3})}{\sqrt{3\pi}}$.

- If $p = 2$, then $\lambda(\underline{a}', p) = -\dfrac{a_2^2}{2} + \dfrac{a_1}{2}$. We deduce that, for all $a_2 \in \mathbb{C}$,

$$C_0(a_2^2, a_2, \frac{3}{4}) = -e^{-\frac{i\pi}{3}}.$$

- If $p = 3$, then $\lambda(\underline{a}', p) = \dfrac{a_2^3}{12} - \dfrac{a_2 a_1}{3} + \dfrac{1}{3}$. Thus, for all $a_2 \in \mathbb{C}^\star$,

$$C_0(\frac{4 + a_2^3}{4a_2}, a_2, 2) = e^{-\frac{i\pi}{3}}.$$

Since $\lambda(\underline{a}', p)$ can be computed exactly for all fixed $p \in \mathbb{N}^\star$, it is natural to try to get more informations from equation (98). The result is a little bit disappointing, as we now explain.

We assume that

$$\widetilde{L}_0(\underline{a}', p)\widetilde{L}_0(\omega.\underline{a}', p)\widetilde{L}_0(\omega^2.\underline{a}', p) \neq 0. \tag{101}$$

We note that, by remark 7.2,

$$\widetilde{L}_0(0,p) = e^{-\frac{i\pi}{3}} 3^{\frac{p}{3}} e^{i\pi\frac{p}{3}} \frac{\Gamma(\frac{p}{3})}{\sqrt{3\pi}}$$

so that hypothesis (101) is valid for $\underline{a}'$ in a neighbourhood of the origin. We can write (98) as

$$y(\underline{a}',p) + \omega^p y(\omega.\underline{a}',p) + \omega^{2p} y(\omega^2.\underline{a}',p) = -\pi p \omega^{1-\frac{p}{2}} \lambda(\underline{a}',p) \tag{102}$$

with

$$y(\underline{a}',p) = \frac{1}{\widetilde{L}_0(\underline{a}',p)\widetilde{L}_0(\omega.\underline{a}',p)}.$$

Equation (102) can be thought of as a non-homogenous second order linear q-difference equation. Unfortunately, we are in the worst situation, when q is a root of unity, so that solving (102) gives very little information. Indeed, we first observe that $-\frac{\pi p}{3}\omega^{1-\frac{p}{2}}\lambda(\underline{a}',p)$ is a particular solution of (102), because $\lambda(\omega.\underline{a}',p) = \omega^{-p}\lambda(\underline{a}',p)$. Therefore, by linearity of (102), one only needs to solve the homogenous equation

$$y(\underline{a}',p) + \omega^p y(\omega.\underline{a}',p) + \omega^{2p} y(\omega^2.\underline{a}',p) = 0 \tag{103}$$

in the space $\mathbb{C}\{a_1,a_2\}$. Writing $y(\underline{a}',p) = \sum_{k,l=0}^{\infty} b_{k,l} a_1^k a_2^l$, equation (103) is equivalent to (since $\omega^3 = e^{2i\pi}$):

$$\sum_{k,l=0}^{\infty} (1 + \omega^{p+k+2l} + \omega^{2p+2k+l}) b_{k,l} a_1^k a_2^l = 0.$$

Thus, $y \in \mathbb{C}\{a_1,a_2\}$ is solution of (103) provided that $b_{k,l} = 0$ when $p+k+2l = 0 \mod 3$. This corresponds to a vector-space of infinite dimension !

To end this subsection, we mention [36] for the numerical computations of the 0∞ connection matrices.

7.3.2 the case $m = 4$

When $m = 4$, we have $r(\underline{a}) = -\frac{1}{2} - \frac{1}{2}a_2 + \frac{1}{8}a_1^2$ so that, by theorem 2.7,

$$\widetilde{C}_0(\underline{a}) = i e^{-\frac{i\pi}{2}(a_2 - \frac{1}{4}a_1^2)}.$$

Also, by corollary 6.3 :

$$\begin{gathered} C_0(\underline{a})C_1(\underline{a})C_2(\underline{a})C_3(\underline{a}) + \widetilde{C}_0(\underline{a})C_2(\underline{a})C_3(\underline{a}) + \widetilde{C}_1(\underline{a})C_0(\underline{a})C_3(\underline{a}) + \\ \widetilde{C}_2(\underline{a})C_0(\underline{a})C_1(\underline{a}) + \widetilde{C}_3(\underline{a})C_1(\underline{a})C_2(\underline{a}) + \widetilde{C}_0(\underline{a})\widetilde{C}_2(\underline{a}) + \widetilde{C}_1(\underline{a})\widetilde{C}_3(\underline{a}) = -2\cos(\pi p). \end{gathered} \tag{104}$$

We already know, by corollary 7.6 applied with $n = 2$, that

$$C_0(0,a_2,0,a_4) = e^{-\frac{i\pi}{4}(a_2+1)} 2^{-\frac{a_2}{2}+1} \frac{\Gamma(\frac{p}{4} + \frac{a_2}{4} + \frac{1}{2})}{\Gamma(\frac{p}{4} - \frac{a_2}{4} + \frac{1}{2})} \cos\left(\frac{\pi}{4}(p + a_2)\right) \tag{105}$$

with $p = (1 + 4a_4)^{\frac{1}{2}}$.

Propositions similar to proposition 7.7 can be obtained for every $m \geq 3$. In particular for $m = 4$, we show what happens for values of $\underline{a}$ such that $a_4 = \dfrac{p^2-1}{4}$ with $p \in \mathbb{N}^\star$ and $\lambda(\underline{a}', p) = 0$. Evaluating the last row of the product $\mathfrak{S}_0(\underline{a})\mathfrak{S}_1(\underline{a}) \cdots \mathfrak{S}_3(\underline{a})$ and applying corollary 6.4, one gets

$$\begin{pmatrix} \cdots & \cdots \\ \left(C_1(\underline{a})C_2(\underline{a}) + \widetilde{C}_1(\underline{a})\right) C_3(\underline{a}) + C_1(\underline{a})\widetilde{C}_2(\underline{a}) & \left(C_1(\underline{a})C_2(\underline{a}) + \widetilde{C}_1(\underline{a})\right) \widetilde{C}_3(\underline{a}) \end{pmatrix}$$
$$= (-1)^{p+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $\omega = e^{\frac{i\pi}{2}}$ so that $\widetilde{C}_2(\underline{a}) = \widetilde{C}_0(\omega^2.\underline{a}) = \widetilde{C}_0(\underline{a})$. We thus get

$$C_0(\underline{a})C_1(\underline{a}) + \widetilde{C}_0(\underline{a}) = (-1)^{p+1}\widetilde{C}_0^{-1}(\underline{a})$$

and

$$\left(C_0(\underline{a})C_1(\underline{a}) + \widetilde{C}_0(\underline{a})\right) C_2(\underline{a}) + C_0(\underline{a})\widetilde{C}_1(\underline{a}) = 0.$$

Since $\widetilde{C}_0(\underline{a})\widetilde{C}_1(\underline{a}) = -1$, we obtain

$$C_0(\underline{a}) = (-1)^{p+1}C_2(\underline{a}), \qquad C_0(\underline{a})C_1(\underline{a}) = (-1)^{p+1}\widetilde{C}_0^{-1}(\underline{a}) - \widetilde{C}_0(\underline{a}).$$

Computing $\lambda(\underline{a}', p)$, one obtains for instance:

- If $p = 1$, then $\lambda(\underline{a}', p) = a_3$. Therefore, for all $(a_1, a_2) \in \mathbb{C}^2$,

  $$C_0(a_1, a_2, 0, 0) = C_2(a_1, a_2, 0, 0),$$
  $$C_0(a_1, a_2, 0, 0)C_1(a_1, a_2, 0, 0) = -2i \cos\left(\frac{\pi}{2}(a_2 - \frac{1}{4}a_1^2)\right).$$

  This case corresponds to the Weber equation. By a Tschirnhaus transformation, one can use equation (105) to get

  $$C_0(a_1, a_2, 0, 0) = \\ 2e^{-\frac{i\pi}{4}} e^{-i\pi\frac{1}{4}(a_2 - \frac{1}{4}a_1^2)} 2^{-\frac{a_2}{2} + \frac{1}{8}a_1^2} \frac{\Gamma(\frac{a_2}{4} - \frac{1}{16}a_1^2 + \frac{3}{4})}{\Gamma(-\frac{a_2}{4} + \frac{1}{16}a_1^2 + \frac{3}{4})} \cos\left(\frac{\pi}{4}(a_2 - \frac{1}{4}a_1^2 + 1)\right).$$

  By the Euler reflection formula and the duplication formula for the Gamma function, one gets the usual well-known formula (cf. [35]).

  Comparing this result with (105), it is tempting to conjecture that:

  $$C_0(a_1, a_2, 0, a_4) = \\ e^{-\frac{i\pi}{4}(a_2 - \frac{1}{4}a_1^2 + 1)} 2^{-\frac{a_2}{2} + \frac{1}{8}a_1^2 + 1} \frac{\Gamma(\frac{p}{4} + \frac{a_2}{4} - \frac{1}{16}a_1^2 + \frac{1}{2})}{\Gamma(\frac{p}{4} - \frac{a_2}{4} + \frac{1}{16}a_1^2 + \frac{1}{2})} \cos\left(\frac{\pi}{4}(p + a_2 - \frac{1}{4}a_1^2)\right).$$

  (This satisfies the functional relation (104)).

- If $p = 2$, then $\lambda(\underline{a}', p) = -\dfrac{a_3^2}{2} + \dfrac{a_2}{2}$. We deduce that, for all $(a_1, a_3) \in \mathbb{C}^2$,

  $$C_0(a_1, a_3^2, a_3, \tfrac{3}{4}) = -C_2(a_1, a_3^2, a_3, \tfrac{3}{4}),$$
  $$C_0(a_1, a_3^2, a_3, \tfrac{3}{4})C_1(a_1, a_3^2, a_3, \tfrac{3}{4}) = -2\sin\left(\tfrac{\pi}{2}(a_3^2 - \tfrac{1}{4}a_1^2)\right).$$

- If $p=3$, then $\lambda(\underline{a}',p)=\dfrac{a_3^3}{12}-\dfrac{a_2a_3}{3}+\dfrac{a_1}{3}$. Thus, for all $(a_2,a_3)\in\mathbb{C}^2$,

$$C_0(-\tfrac{a_3^3}{4}+a_2a_3,a_2,a_3,2)=C_2(-\tfrac{a_3^3}{4}+a_2a_3,a_2,a_3,2),$$

$$C_0(-\tfrac{a_3^3}{4}+a_2a_3,a_2,a_3,2)C_1(-\tfrac{a_3^3}{4}+a_2a_3,a_2,a_3,2)= \\ -2i\cos\left(\tfrac{\pi}{2}(a_2-\tfrac{1}{4}(-\tfrac{a_3^3}{4}+a_2a_3)^2)\right).$$

A Using special functions

A.1 Example 1 : a normal form of Heun's equation

We consider the equation

$$(\mathfrak{E}_1)\qquad x^2\Phi''=(x+a)\Phi.$$

This is the simplest case, when $m=1$. In this case, the Stokes-Sibuya connection matrix $\mathfrak{S}_0$ is given by proposition 7.1. This proposition provides also the 0∞ connection matrix M_0, up to an odd entire function of $p=(1+4a)^{\frac{1}{2}}$, which we are going to compute here thanks to the fact that the above normal form $(\mathfrak{E}_1)$ of Heun's equation reduces to a modified Bessel equation. Indeed, setting

$$\begin{cases} t=2x^{1/2} \\ \Psi(t)=x^{-1/2}\Phi(x) \end{cases}\tag{106}$$

equation $(\mathfrak{E}_1)$ is converted into the equation :

$$t^2\Psi''(t)+t\Psi'(t)-(t^2+p^2)\Psi(t)=0,\qquad p=(1+4a)^{\frac{1}{2}},\tag{107}$$

which is a modified Bessel equation of parameter p. Thus, we can use the well-known special functions associated with the modified Bessel equation.

We assume that $p=(1+4a)^{\frac{1}{2}}\notin\mathbb{Z}$.

With the notations of theorem 4.2, one easily gets the fundamental system of solutions (f_1,f_2) of $(\mathfrak{E}_1)$ in the form:

$$\begin{cases} f_1(x,p)=\Gamma(p+1)\sqrt{x}I_p(2\sqrt{x}),\quad I_p(t)=\left(\dfrac{t}{2}\right)^p\displaystyle\sum_{n=0}^{+\infty}\frac{1}{n!\Gamma(n+p+1)}\left(\frac{t}{2}\right)^{2n} \\ f_2(x,p)=\Gamma(-p+1)\sqrt{x}I_{-p}(2\sqrt{x}),\quad I_{-p}(t)=\left(\dfrac{t}{2}\right)^{-p}\displaystyle\sum_{n=0}^{+\infty}\frac{1}{n!\Gamma(n-p+1)}\left(\frac{t}{2}\right)^{2n}. \end{cases}\tag{108}$$

where I_p (respectively I_{-p}) is the modified Bessel function (or Bessel function of imaginary argument) of order p (respectively of order $-p$), see [30].

Remark A.1. We recall that the functions I_p and I_{-p} are very closely connected to the Bessel functions of the first kind J_p and J_{-p} by (see [30]) :

$$\begin{cases} I_p(t)=e^{-\frac{i\pi p}{2}}J_p(it) \\ I_{-p}(t)=e^{\frac{i\pi p}{2}}J_{-p}(it) \end{cases}$$

Now, thanks to theorem 2.1, there exists an unique solution Φ_0 of $(\mathfrak{E}_1)$, asymptotic at infinity to $T\Phi_0(x,a) = e^{-2\sqrt{x}}x^{\frac{1}{4}}\phi_0(x,a)$ with $\phi_0 \in \mathbb{C}[a][[x^{-\frac{1}{2}}]]$ in the sector $-3\pi < arg(x) < 3\pi$. Precisely

$$T\Phi_0(x,a) = e^{-2\sqrt{x}}x^{\frac{1}{4}}\left(1+\sum_{n=1}^{+\infty}\frac{(4p^2-1)\dots(4p^2-(2n-1)^2)}{n!16^n x^{\frac{n}{2}}}\right) \tag{109}$$
$$\text{in } \Sigma_0 = \{|\arg(x)| < 3\pi\}.$$

Also, by lemma 2.3 and theorem 2.7, we have a fundamental system of solutions (Φ_0, Φ_1) of $(\mathfrak{E}_1)$, where Φ_1 is characterized by the following asymptotic expansion at infinity :

$$T\Phi_1(x,a) = e^{2\sqrt{x}}\omega^{\frac{1}{4}}x^{\frac{1}{4}}\left(1+\sum_{n=1}^{+\infty}(-1)^n\frac{(4p^2-1)\dots(4p^2-(2n-1)^2)}{n!16^n x^{\frac{n}{2}}}\right) \tag{110}$$
$$\text{in } \Sigma_1 = \{|\arg(x)+2\pi| < 3\pi\}$$

where $\omega = e^{2i\pi}$. As we shall see, these functions Φ_0 and Φ_1 can be expressed with the MacDonald functions $K_p^{(1)}$ and $K_p^{(2)}$.

The MacDonald functions $K_p^{(1)}(t)$ and $K_p^{(2)}(t)$ are analytic functions in the variable t for t not equal to zero; they are linearly independent ($W(K_p^{(1)}, K_p^{(2)}) = \frac{\pi}{t}$) and solutions of the modified Bessel equation (107). These functions are derived from the Hankel functions by the following relationships :

$$\begin{cases} K_p^{(1)}(t) = \dfrac{1}{2}i\pi e^{\frac{pi\pi}{2}}H_p^{(1)}(it) \\ \\ K_p^{(2)}(t) = \dfrac{1}{2}\pi e^{-\frac{pi\pi}{2}}H_p^{(2)}(it). \end{cases} \tag{111}$$

Furthermore, they admit respectively an asymptotic expansion $TK_p^{(1)}$ and $TK_p^{(2)}$ when t tends to infinity of the form (see [30]) :

$$TK_p^{(1)}(t) = \left(\frac{\pi}{2t}\right)^{\frac{1}{2}}e^{-t}\left(1+\sum_{n=1}^{+\infty}\frac{(4p^2-1)\cdots(4p^2-(2n-1)^2)}{n!8^n t^n}\right) \tag{112}$$

$$\text{in } \widetilde{\Sigma}_0 = \{|\arg(t)| < \tfrac{3\pi}{2}\}$$

and

$$TK_p^{(2)}(t) = \left(\frac{\pi}{2t}\right)^{\frac{1}{2}}e^{t}\left(1+\sum_{n=1}^{+\infty}(-1)^n\frac{(4p^2-1)\cdots(4p^2-(2n-1)^2)}{n!8^n t^n}\right) \tag{113}$$

$$\text{in } \widetilde{\Sigma}_1 = \{|\arg(t)+\pi| < \tfrac{3\pi}{2}\}.$$

Using (106) and (107), we deduce by uniqueness of Φ_0 (resp. Φ_1), comparing (112) (resp. (113)) with (109) (resp. (110)), that

$$\Phi_0(x,a) = \frac{2}{\sqrt{\pi}}\sqrt{x}K_p^{(1)}(2\sqrt{x}) \tag{114}$$

and

$$\Phi_1(x,a) = \frac{2}{\sqrt{\pi}}\omega^{\frac{1}{4}}\sqrt{x}K_p^{(2)}(2\sqrt{x}). \tag{115}$$

Recalling the connection formulas (see [30]),

$$\begin{cases} J_p(t) = \dfrac{1}{2}(H_p^{(1)}(t) + H_p^{(2)}(t)) \\ \\ J_{-p}(t) = \dfrac{1}{2}(e^{i\pi p}H_p^{(1)}(t) + e^{-i\pi p}H_p^{(2)}(t)), \end{cases}$$

we deduce with remark A.1 and (111):

$$\begin{cases} I_p(t) = \dfrac{e^{-i\pi p}}{i\pi}K_p^{(1)}(t) + \dfrac{1}{\pi}K_p^{(2)}(t) \\ \\ I_{-p}(t) = \dfrac{e^{i\pi p}}{i\pi}K_p^{(1)}(t) + \dfrac{1}{\pi}K_p^{(2)}(t). \end{cases} \tag{116}$$

Putting (108), (114), (115) and (116) together, we obtain

$$\begin{pmatrix} f_1 \\ \\ f_2 \end{pmatrix}(x,p) = \begin{pmatrix} -i\dfrac{\Gamma(p+1)}{2\sqrt{\pi}}e^{-i\pi p} & -i\dfrac{\Gamma(p+1)}{2\sqrt{\pi}} \\ \\ -i\dfrac{\Gamma(-p+1)}{2\sqrt{\pi}}e^{i\pi p} & -i\dfrac{\Gamma(-p+1)}{2\sqrt{\pi}} \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix}(x,a) \tag{117}$$

where the matrix on the right-hand side of this equality is the inverse of the 0∞ connection matrix M_1 (cf. formula (57)). By proposition 5.2 we deduce:

$$M_0(p) = \begin{pmatrix} -e^{-i\pi p}\dfrac{\Gamma(-p)}{\sqrt{\pi}} & -e^{i\pi p}\dfrac{\Gamma(p)}{\sqrt{\pi}} \\ \\ \dfrac{\Gamma(-p)}{\sqrt{\pi}} & \dfrac{\Gamma(p)}{\sqrt{\pi}} \end{pmatrix}.$$

Remark A.2. This result is consistent with proposition 7.1 and remark 7.2.

A.2 Example 2 : a normal form of Whittaker's equation

We now focus on the equation

$$(\mathfrak{E}_2) \qquad x^2\Phi'' = (x^2 + a_1x + a_2)\Phi.$$

This equation reduces to the Whittaker equation. Indeed, the transformation

$$\begin{cases} x = \frac{t}{2} \\ \phi(t) = \Phi(x) \end{cases}$$

converts equation $(\mathfrak{E}_2)$ into :

$$\phi''(t) = (\frac{1}{4} + \frac{a_1}{2t} + \frac{a_2}{t^2})\phi(t)$$

which is the Whittaker equation of parameters

$$k = -\frac{a_1}{2} \quad \text{and} \quad n = \frac{p}{2} = (\frac{1}{4} + a_2)^{\frac{1}{2}}.$$

In what follows, we shall make a heavy use of the known properties of the special functions associated with the Whittaker equation, see for instance [30].

A.2.1 Study near the origin

We assume here that $p = (1+4a_2)^{\frac{1}{2}} \notin \mathbb{Z}$, i.e $2n \notin \mathbb{Z}$. The fundamental system of solutions (f_1, f_2) of theorem 4.2 can be written as follows:

$$\left\{\begin{array}{c} f_1(x,a_1,p) = 2^{-n-\frac{1}{2}} M_{k,n}(2x) \quad \text{where} \\ M_{k,n}(t) = e^{-\frac{t}{2}} t^{n+\frac{1}{2}} M(n-k+\dfrac{1}{2}, 2n+1, t), \quad M(\alpha,c,t) = \displaystyle\sum_{s=0}^{+\infty} \frac{(\alpha)_s}{(c)_s}\frac{t^s}{s!} \\ \\ f_2(x,a_1,p) = 2^{n-\frac{1}{2}} N_{k,n}(2x) \quad \text{where} \\ N_{k,n}(t) = e^{-\frac{t}{2}} t^{n+\frac{1}{2}} N(n-k+\dfrac{1}{2}, 2n+1, t), \quad N(\alpha,c,t) = t^{1-c} M(1+\alpha-c, 2-c, t). \end{array}\right. \tag{118}$$

with the Pochhammer's notation:

$$\left\{\begin{array}{l} (\alpha)_0 = 1 \\ (\alpha)_s = \alpha(\alpha+1)\dots(\alpha+s-1). \end{array}\right.$$

Remark A.3. The function $M_{k,n}$ is called a Whittaker function and $M(\alpha, c, t)$ is called the Kummer function (which is an entire function in t).

A.2.2 Study at infinity

In theorem 2.1, the solution Φ_0 of $(\mathfrak{E}_2)$ Φ_0 can be characterized by its asymptotics. Since

$$r(\underline{a}) = -\frac{a_1}{2} = k \quad \text{and} \quad \omega = e^{i\pi},$$

we have $T\Phi_0(x,\underline{a}) = x^k e^{-x} \phi_0(x,\underline{a})$ with $\phi_0(x,\underline{a}) \in \mathbb{C}[a_1,a_2][[x^{-1}]]$ with constant term 1, in the sector $-\frac{3\pi}{2} < arg(x) < \frac{3\pi}{2}$.
In the same way, Φ_1 is characterized by its asymptotics, $T\Phi_1(x,\underline{a}) = \omega^{-k} x^{-k} e^x \phi_1(x,\underline{a})$ where $\phi_1(x,\underline{a}) \in \mathbb{C}[a_1,a_2][[x^{-1}]]$ with constant term 1, in the sector $-\frac{3\pi}{2} < arg(x) + \pi < \frac{3\pi}{2}$.
In fact, these two functions Φ_0 and Φ_1 can be expressed in terms of the functions U and V of the confluent hypergeometric equation:

$$t\frac{d^2 f}{dt^2} + (c-t)\frac{df}{dt} - \alpha f = 0.$$

Proposition A.4.

$$\left\{\begin{array}{l} \Phi_0(x,\underline{a}) = 2^{-k} W_{k,n}(2x) \quad \textit{where} \quad W_{k,n}(t) = e^{-\frac{t}{2}} t^{n+\frac{1}{2}} U(n-k+\dfrac{1}{2}, 2n+1, t) \\ \textit{and} \quad U(\alpha,c,t) \sim t^{-\alpha} \displaystyle\sum_{s=0}^{+\infty} \frac{(-1)^s (1+\alpha-c)_s}{s! t^s} \ \textit{in the sector} \ \mid arg(t) \mid < \frac{3\pi}{2} \\ \\ \Phi_1(x,\underline{a}) = i2^k e^{n\pi i} V_{k,n}(2x) \quad \textit{where} \quad V_{k,n}(t) = e^{-\frac{t}{2}} t^{n+\frac{1}{2}} V(n-k+\dfrac{1}{2}, 2n+1, t) \\ \textit{and} \quad V(\alpha,c,t) \sim e^t (e^{i\pi} t)^{\alpha-c} \displaystyle\sum_{s=0}^{+\infty} \frac{(c-\alpha)_s (1-\alpha)_s}{s! t^s} \ \textit{in the sector} \ \mid arg(t) + \pi \mid < \frac{3\pi}{2} \end{array}\right.$$

Remark A.5. The function $W_{k,n}$ is also called a Whittaker function, see [30].

A.2.3 Connection formulas

We recall the following connection formula (see [30]):

$$M(\alpha,c,t)=\Gamma(c)\Big(\frac{e^{-\alpha\pi i}}{\Gamma(c-\alpha)}U(\alpha,c,t)+\frac{e^{(c-\alpha)\pi i}}{\Gamma(\alpha)}V(\alpha,c,t)\Big)$$

Therefore,

$$M_{k,n}(2x)=\frac{-ie^{(k-n)\pi i}\Gamma(2n+1)}{\Gamma(n+k+\frac{1}{2})}W_{k,n}(2x)+\frac{ie^{(k+n)\pi i}\Gamma(2n+1)}{\Gamma(n-k+\frac{1}{2})}V_{k,n}(2x)$$

which means that

$$f_1(x,a_1,p)=-ie^{i\pi(k-n)}\frac{2^{k-n}\Gamma(2n+1)}{\sqrt{2}\Gamma(n+k+\frac{1}{2})}\Phi_0(x,\underline{a})+e^{i\pi k}\frac{2^{-k-n}\Gamma(2n+1)}{\sqrt{2}\Gamma(n-k+\frac{1}{2})}\Phi_1(x,\underline{a}). \tag{119}$$

Furthermore, thanks to the connection formula (see [30])

$$U(\alpha,c,t)=\frac{\Gamma(1-c)}{\Gamma(1+\alpha-c)}M(\alpha,c,t)-\frac{\Gamma(c)\Gamma(1-c)}{\Gamma(\alpha)\Gamma(2-c)}N(\alpha,c,t),$$

we deduce that :

$$N(\alpha,c,t)=\Big(\frac{\Gamma(\alpha)\Gamma(2-c)e^{-\alpha\pi i}}{\Gamma(1+\alpha-c)\Gamma(c-\alpha)}-\frac{\Gamma(\alpha)\Gamma(2-c)}{\Gamma(c)\Gamma(1-c)}\Big)U(\alpha,c,t)+\frac{e^{(c-\alpha)\pi i}\Gamma(2-c)}{\Gamma(1+\alpha-c)}V(\alpha,c,t)$$

so that :

$$N_{k,n}(2x)=\Big(\frac{\Gamma(n-k+\frac{1}{2})\Gamma(1-2n)e^{-(n-k+\frac{1}{2})\pi i}}{\Gamma(-n-k+\frac{1}{2})\Gamma(n+k+\frac{1}{2})}-\frac{\Gamma(n-k+\frac{1}{2})\Gamma(1-2n)}{\Gamma(2n+1)\Gamma(-2n)}\Big)W_{k,n}(2x)$$
$$+\Big(\frac{e^{(n+k+\frac{1}{2})\pi i}\Gamma(1-2n)}{\Gamma(-n-k+\frac{1}{2})}\Big)V_{k,n}(2x)$$

i.e :

$$f_2(x,a_1,p)=$$
$$\Big(\frac{-i2^{k+n}e^{(k-n)\pi i}\Gamma(1-2n)\Gamma(n-k+\frac{1}{2})}{\sqrt{2}\Gamma(n+k+\frac{1}{2})\Gamma(-n-k+\frac{1}{2})}-\frac{2^{k+n}\Gamma(1-2n)\Gamma(n-k+\frac{1}{2})}{\sqrt{2}\Gamma(2n+1)\Gamma(-2n)}\Big)\Phi_0(x,\underline{a})$$
$$+\Big(\frac{2^{n-k}e^{k\pi i}\Gamma(1-2n)}{\sqrt{2}\Gamma(-n-k+\frac{1}{2})}\Big)\Phi_1(x,\underline{a}).$$

which reads also:

$$f_2(x,a_1,p)=-ie^{i\pi(n+k)}\frac{2^{k+n}\Gamma(1-2n)}{\sqrt{2}\Gamma(-n+k+\frac{1}{2})}\Phi_0(x,\underline{a})+e^{i\pi k}\frac{2^{n-k}\Gamma(1-2n)}{\sqrt{2}\Gamma(-n-k+\frac{1}{2})}\Phi_1(x,\underline{a}). \tag{120}$$

Formulas (119) and (120) yield the inverse of the 0∞ connection matrix M_1. Going back to the variables p and a_1, one gets:

$$M_1^{-1}(a_1,p) = \begin{pmatrix} -ie^{i\pi(-\frac{a_1}{2}-\frac{p}{2})}\dfrac{2^{-\frac{a_1}{2}-\frac{p}{2}}\Gamma(p+1)}{\sqrt{2}\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} & e^{-i\pi\frac{a_1}{2}}\dfrac{2^{\frac{a_1}{2}-\frac{p}{2}}\Gamma(p+1)}{\sqrt{2}\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})} \\ \\ -ie^{i\pi(-\frac{a_1}{2}+\frac{p}{2})}\dfrac{2^{-\frac{a_1}{2}+\frac{p}{2}}\Gamma(1-p)}{\sqrt{2}\Gamma(-\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} & e^{-i\pi\frac{a_1}{2}}\dfrac{2^{\frac{a_1}{2}+\frac{p}{2}}\Gamma(1-p)}{\sqrt{2}\Gamma(-\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})} \end{pmatrix}. \tag{121}$$

Using proposition 5.2 we deduce that

$$M_0(a_1,p) = \begin{pmatrix} -ie^{-i\pi\frac{p}{2}}\dfrac{2^{-\frac{a_1}{2}+\frac{p}{2}+1}\Gamma(-p)}{\sqrt{2}\Gamma(-\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} & -ie^{i\pi\frac{p}{2}}\dfrac{2^{-\frac{a_1}{2}-\frac{p}{2}+1}\Gamma(p)}{\sqrt{2}\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})} \\ \\ \dfrac{2^{\frac{a_1}{2}+\frac{p}{2}+1}\Gamma(-p)}{\sqrt{2}\Gamma(-\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})} & \dfrac{2^{\frac{a_1}{2}-\frac{p}{2}+1}\Gamma(p)}{\sqrt{2}\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})} \end{pmatrix}. \tag{122}$$

Remark A.6. This result is consistent with theorem 5.3. Also, with the notations of theorem 5.3, we have found

$$\widetilde{L}_0(a_1,p) = -ie^{i\pi\frac{p}{2}}\frac{2^{-\frac{a_1}{2}-\frac{p}{2}+1}\Gamma(p)}{\sqrt{2}\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})}.$$

In particular, when $a_1 = 0$ we get, using the Legendre duplication formula for the Gamma function:

$$\widetilde{L}_0(0,p) = 2^{\frac{p}{2}}e^{i\pi\frac{p}{2}}\frac{(-i)}{\sqrt{2\pi}}\Gamma(\frac{p}{2}),$$

a result which agrees also with proposition 7.1 and remark 7.2.

By formula (65) of theorem 6.2, we have $\mathfrak{S}_0(\underline{a}) = M_0(a_1,p)M_1^{-1}(a_1,p)$ and the result extends to $2n \in \mathbb{Z}$ by analytic continuation, since $\mathfrak{S}_0$ is entire. We eventually get:

Proposition A.7. *We assume $m = 2$. Then, for all $\underline{a} = (a_1,a_2) \in \mathbb{C}^2$, the Stokes-Sibuya connection matrix $\mathfrak{S}_0$ is given by*

$$\mathfrak{S}_0(\underline{a}) = \begin{pmatrix} -2ie^{-i\pi\frac{a_1}{2}}2^{-a_1}\dfrac{\Gamma(\frac{p}{2}+\frac{a_1}{2}+\frac{1}{2})}{\Gamma(\frac{p}{2}-\frac{a_1}{2}+\frac{1}{2})}\cos\left((\dfrac{p}{2}+\dfrac{a_1}{2})\pi\right) & e^{-i\pi a_1} \\ \\ 1 & 0 \end{pmatrix}$$

where

$$p = (1+4a_2)^{\frac{1}{2}}.$$

Moreover, when $p \notin \mathbb{Z}$, the 0∞ connection matrix M_0 is given by formula (122).

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